Math 53 Dis 108/111 Midterm 2 Worksheet (10/30)

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1 Lagrange Multipliers

This section will probably have 1-2 questions.

Problem 1. Find the maximum and minimum values of $f(x, y, z) = x^2 + z^2$ subject to $x^4 + y^4 + z^4 \leqslant 16$.

Problem 2. [Textbook 14.8.34] Using Lagrange multipliers, find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.

Problem 3. [Textbook 14.8.30] Use Lagrange multipliers to prove that the largest area a triangle of perimeter p can have is the area of the equilateral triangle with perimeter p. You may use Heron's formula for the area of a triangle:

$$A = \sqrt{s(s-x)(s-y)(s-z)}$$

where $s = p/2$ and x, y, z are the three side lenghts.

2 Multivariable Integration

This section will probably have 4-5 questions.

Problem 0. (Warm-up). Consider the region $0 \geqslant x \geqslant y \geqslant -1$. Let $f(x, y)$ be an arbitrary function. Set up the integral of f over the region in both orders of integration.

Problem 1. Compute

$$\int_0^1 \int_x^1 x\sqrt{1+y^3}\,\mathrm{d}y\,\mathrm{d}x.$$

Problem 2. [Daniel Tataru] Evaluate the integral

$$\int_0^2 \int_{y-1}^1 \sqrt{x^2+2x+2}\,\mathrm{d}x\,\mathrm{d}y.$$

Problem 3. Consider the region bounded by $z = \sqrt[4]{x}$, $z = \sqrt[2]{x}$, $y = z^2$, and $y = 0$. Find the volume.

Problem 4. Consider the region D bounded by $z = x$, $z = -x$, and $z = 1 - y^2$. Compute

$$\iiint_D \sqrt{1-z}\,\mathrm{d}V$$

Problem 5. [Daniel Tataru] Consider $f(x, y, z) = 6z$ and the region below the cone $z^2 = x^2 + y^2$ and above the parabola $z = x^2 + y^2$. Compute

$$\iiint_D f\,\mathrm{d}V.$$

Problem 6. Compute the volume inside the region $z^2 + y^2 = 1 + x^2$ between $x = 1$ and $x = -1$.

Problem 7. Consider the region $a^2 \leqslant x^2 + y^2 + z^2 \leqslant b^2$ (where $a < b$ are positive real numbers) and $z < 0$. Compute the average distance from the origin.

Problem 8. [Nikhil Srivastava] Consider the parallelogram P with vertices $(0,0)$, $(1,1)$, $(2,-1)$, and $(3,0)$. Evaluate

$$\iint\limits_P (x+2y)^2 e^{x-y}\,\mathrm{d}A$$

using the change of variables $u = \frac{x+2y}{3}$ and $v = \frac{x-y}{3}$.

Problem 9. Compute the area of the weird ellipse $(3x-2y)^2 + 3x^2 \leqslant 3$.

3 Line Integrals

This section will probably have 2-3 questions.

Problem 0. Explain in words why all mixed partials have to be equal for a vector field to be conservative. What other conditions are there for a vector field to be conservative?

Problem 1. [Nikhil Srivastava] Consider the vector field

$$\mathbf{F} = \left\langle 4x\ln(y), \frac{2x^2-1}{y} \right\rangle.$$

(a) Show the vector field is conservative.

(b) Find a function f such that $f = \nabla\mathbf{F}$.

Problem 2. Suppose your shower curtain traces out the curve $y = \cos(10x)/10 + \sin(2x)$ on the floor and that the height of the shower curtain at (x,y) is y. Set up the integral to compute the area of the shower curtain.

Problem 3. Consider the line $x = \cos(t)$ and $y = \cos(2t)$ for $t \in [0,\pi]$. Evaluate

$$\int_C \frac{1}{x}\,\mathrm{d}y + \mathrm{d}x.$$

Problem 4. Let $\mathbf{F} = \left\langle 2xy + \frac{1}{y}, 3 + x^2 - \frac{x}{y^2}, 1 \right\rangle$. Compute $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ where C is the curve $x(t) = \cos(t) - \frac{1}{7}\sin(8t)$, $y(t) = 4 + e^t\sin(100t)$, and $z = \frac{t}{\pi}e^{\cos^2 t}$ for $t \in [0,2\pi]$.

Problem 6. [Paul's Online Notes] Compute

$$\int_C yx^2\,\mathrm{d}x - x^2\,\mathrm{d}y$$

where C is the left half of a unit circle (forming a closed loop using the y-axis).

Problem 7. [Daniel Tataru]

(a) State Green's theorem on the region $1 \leqslant x^2 + y^2$ and $x^2 + 4y^2 \leqslant 16$.

(b) Is the vector field

$$\mathbf{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

conservative?

(c) Compute

$$\oint\limits_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$$

where $C = x^2 + 4y^2 = 16$.