Math 53 Dis 108/111 Worksheet 10 (10/16)

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Problem 0. (Warm-up). Consider the triangle with vertices (0,-2), (0,1) and (1,-2). Write the integral in both orders of integration.

Problem 1: Polar Integrals. Recall the formula for double integrals in polar coordinates

$$\iint_D f(r,\theta) r \, \mathrm{d}r \, \mathrm{d}\theta$$

Note that we almost always integrate $\mathrm{d}r\,\mathrm{d}\theta$: it's a lot more natural to find bounds on r that depend on θ than bounds on θ that depend on r

I. Consider the annulus with outer radius 3 and inner radius 1 between the angles $\pi/6$ and $2\pi/3$.

  (a) Calculate the area without using calculus.

  (b) Calculate the area using 1-variable calculus.

  (c) Calculate the area using 2-variable calculus.

II. Integrate $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 \leqslant 4$.

III. Integrate $f(x,y) = xy$ on the circle $x^2 + y^2 \leqslant 4$.

IV. Integrate $f(x,y) = 9x^2 + 2y^2$ on the ellipse $9x^2 + y^2 \leqslant 1$. [Hint: write it out in rectangular coordinates, then do a change of variables, then write it out in polar coordinates]

V. (Textbook 15.3.39) Use polar coordinates to combine the sum

$$\int_{1/\sqrt{2}}^{1} \int_{\sqrt{1-x^2}}^{x} xy \, \mathrm{d}y \, \mathrm{d}x + \int_{1}^{\sqrt{2}} \int_{0}^{x} xy \, \mathrm{d}y \, \mathrm{d}x + \int_{\sqrt{2}}^{2} \int_{0}^{\sqrt{4-x^2}} xy \, \mathrm{d}y \, \mathrm{d}x$$

into one integral and then evaluate the integral.

VI. (Classical) Use polar integration to evaluate

$$\int_{-\infty}^{\infty} e^{-x^2} \, \mathrm{d}x.$$

You may use that

$$\left(\int_{-\infty}^{\infty} e^{-x^2} \, \mathrm{d}x\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} \, \mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-y^2} \, \mathrm{d}y\right).$$

Problem 2: Triple Integrals.

I. Consider the triangular prism with vertices at $(0,0,0), (1,0,0), (0,2,0), (0,2,1)$.

  (a) Calculate the volume without using calculus.

  (b) Calculate the volume using a double integral.

  (c) Calculate the volume using a triple integral. Set up the integral in at least 2 different orders of integration.

II. Consider the domain bounded by $y = \cos(x)$, $y = \sin(x)$, $x = 0$ and $z = y$. What is the volume? What is the integral of $f(x,y,z) = x/y$ on this region?

Problem 3: Cylindrical Integrals. Recall the formula for cylindrical integrals

$$\int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z) r \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

(where we are using that $x = r\cos\theta$ and $y = r\sin\theta$).

I. Draw the region defined by

$$\int_{0}^{\pi/2}\int_{1}^{2}\int_{4-r^2}^{4} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$$

II. Evaluate the volume of the region inside the cone $z^2 = x^2 + y^2$ and below the plane $z = 4$.

III. Do the same, but this time for the cone $x^2 = y^2 + z^2$ and the surface $x = y^2 + z^2 + 1/4$.

Problem 4: Spherical Integrals. Recall the formula for spherical integrals

$$\int_{a}^{b}\int_{c}^{d}\int_{g(\theta,\varphi)}^{g(\theta,\varphi)} f(\rho,\theta,\varphi)\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi.$$

I. (Textbook 15.8.7,8) Plot the surfaces $\rho\cos\varphi = 1$ and the surface $\cos\varphi = \rho$.

II. (Textbook 15.8.43,43) Compute

$$\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}}\int_{-\sqrt{a^2-y^2-x^2}}^{\sqrt{a^2-y^2-x^2}}(x^2z + y^2z + z^3)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\int_{-2}^{2}\int_{-\sqrt{4^2-x^2}}^{\sqrt{4^2-x^2}}\int_{2-\sqrt{4^2-y^2-x^2}}^{2+\sqrt{4^2-y^2-x^2}}(x^2 + y^2 + z^2)^{3/2}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

Problem 5: Important Applications. The following two applications are important to know.

I. Surface area. To compute the surface area of the graph of $f(x,y)$ over the region D, we use the formula

$$\iint_D \sqrt{1 + f_x^2 + f_y^2}\,\mathrm{d}x\,\mathrm{d}y.$$

Compute the surface area of a unit sphere.

II. Average value of a function over a region (for 2- and 3-d regions). To compute the average value of f over the region D, we use the formula

$$\frac{\iiint_D f(x,y,z)\,\mathrm{d}V}{\iiint_D 1\,\mathrm{d}V}.$$

Compute the average value of $f(x,y) = x^{2/3}y^{1/3}$ over the region $x^2 \leqslant y \leqslant \sqrt{x}$.

Problem 6: Change of Variables. Recall the formula for the change of variables. Suppose we are integrating a function $f(x,y,z)$ over the region D. Then

$$\iiint_D f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x = \iiint_E f(x(u,v,w), y(u,v,w), z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

IMPORTANT: Notice that when we are changing variables from x,y,z to u,v,w we MUST express x,y,z as function of u,v,w and NOT the other way around (in order to take the Jacobian).

I. Use the change of variables $x = \sqrt{2}u - \sqrt{2/3}v$ and $y = \sqrt{2}u + \sqrt{2/3}v$ to evaluate

$$\iint\limits_R x^2 - xy + y^2 \, \mathrm{d}A$$

over the region bounded by the ellipse $x^2 - xy + y^2 = 2$.

II. (Philip Laporte, Textbook 15.9.23) Find

$$\iint\limits_R \frac{x - 2y}{3x - y} dA$$

where R is the region enclosed by the lines $x = 2y = 0$, $x - 2y = 4$, $3x - y = 1$, $3x - y = 2$.

III. (Textbook 15.9.27) Let R be the region $|x| + |y| \leqslant 1$. Compute the integral

$$\iint\limits_R e^{x+y} \, \mathrm{d}A$$

by using a change of variables.