MATH 53 Quiz 9 (12/1) **Name:**

Please write legibly and explain your work clearly. Answers without explanations may receive less (or no) credit.

Problem 1.

(a) (2 points) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field, and let S be a surface with boundary curve C. State Stokes' theorem.

Solution:

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

(b) (2 points) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field, and let D be a 3-dimensional volume with boundary surface S. State the divergence theorem.

Solution:

$$\iiint_D \operatorname{div} \mathbf{F} \, \mathrm{d}V = \iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

Problem 2. (4 points) Suppose $\mathbf{F} = \langle \sin(xy), \cos(xy), z^2 \rangle$. Let S be the unit sphere in $\mathbb{R}^3$ (i.e. $x^2 + y^2 + z^2 = 1$). Compute

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S}.$$

Solution: First apply Stokes' theorem.

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \iiint_{|x| \leqslant 1} \operatorname{div}(\operatorname{curl} \mathbf{F}) \, \mathrm{d}V$$

and now we use that $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$ (which is true of any $\mathbf{F}$, since divergence of curl is zero) to get that

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \iiint_{|x| \leqslant 1} 0 \, \mathrm{d}V = 0.$$

Problem 3. (4 points) Verify that Stokes' theorem holds for the vector field $\mathbf{F} = \langle y, -x, 0 \rangle$ and the part of the plane $x = z$ inside of the cylinder $x^2 + y^2 = 1$.

First let's compute $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

1. We parameterize the boundary curve as $\mathbf{r}(t) = \langle \cos t, \sin t, \cos t \rangle$. (Note that this has the counterclockwise orientation when viewed from above, which will matter for choosing the correct normal to our surface later)

2. $$\begin{aligned} \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t \\ &= \int_0^{2\pi} \mathbf{F}(\cos t, \sin t, \cos t) \cdot \langle -\sin t, \cos t, -\sin t \rangle\,\mathrm{d}t \\ &= \int_0^{2\pi} \langle \sin t, -\cos t, 0 \rangle \cdot \langle -\sin t, \cos t, -\sin t \rangle\,\mathrm{d}t \\ &= \int_0^{2\pi} -\sin^2 t - \cos^2 t\,\mathrm{d}t = -2\pi \end{aligned}$$

Next let's compute $\iint_S \operatorname{curl} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S$. (I will use the version dotted explicitly with the normal).

1. I compute $\operatorname{curl} \mathbf{F} = \langle 0, 0, -2 \rangle$.

2. I want the upwards pointing normal (see part 1 from the boundary integral). Recall that the (non-unit) normal to a plane $ax + by + cz = 0$ is $\langle a, b, c \rangle$, so the normal to this plane is either $\langle 1, 0, -1 \rangle$ or $\langle -1, 0, 1 \rangle$. The one pointing upwards has the positive z-component. After normalizing, I get $\mathbf{n} = \langle -1/\sqrt{2}, 0, 1/\sqrt{2} \rangle$.

3. I need the measure $\mathrm{d}S$. First, I parametrize my surface as $x = x, y = y, z = x$, from which I "compute"
$$dS = |\langle 1, 0, 1 \rangle \times \langle 0, 1, 0 \rangle|\,\mathrm{d}x\,\mathrm{d}y = |\langle -1, 0, 1 \rangle|\,\mathrm{d}x\,\mathrm{d}y = \sqrt{2}\,\mathrm{d}x\,\mathrm{d}y.$$

4. Finally, I evaluate my integral
$$\begin{aligned} \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}S &= \iint_{x^2+y^2 \leqslant 1} \langle 0, 0, -2 \rangle \cdot \langle -1/\sqrt{2}, 0, 1/\sqrt{2} \rangle \sqrt{2}\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_{x^2+y^2 \leqslant 1} (-2/\sqrt{2})\sqrt{2}\,\mathrm{d}x\,\mathrm{d}y \\ &= -2 \iint_{x^2+y^2 \leqslant 1} \mathrm{d}x\,\mathrm{d}y = -2(\pi 1^2) = -2\pi \end{aligned}$$

and so we have verified Stokes' theorem.