MATH 53 Quiz 8 (11/12) **Name:**

Please write legibly and explain your work clearly. Answers without explanations may receive less (or no) credit.

Problem 1. (4 points) Find all $f(x)$ such that the vector field $\mathbf{F} = \langle f(x), y, -z \rangle$ can be written as the curl of another vector field?

Solution: suppose $\mathbf{F} = \text{curl}\mathbf{G}$. Then $\text{div}\mathbf{F} = \text{div}\text{curl}\mathbf{G} = 0$. Thus

$$0 = \nabla \cdot \mathbf{F} = f'(x) + 1 - 1 \implies f(x) = c$$

so f can be any constant function.

Problem 2. (4 points) Parameterize the surface given by the equation $(y - x^2)^2 + z^2 = 1$. [Hint: do you see any circles?]

Solution: there is a circle in the yz slice for each x, centered at $y = x^2$ and $z = 0$. This means we can set

$$x = u, \qquad y = u^2 + \cos\theta, \qquad z = \sin\theta.$$

This parameterization achieves exactly what we described; for each x, we have traced out a circle of radius 1 in y and z, centered at $y = x^2$ and $z = 0$.

Problem 3. (4 points) Set up the integral to compute $\int \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ where S is the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 3$ (with the downward orientation) and $\mathbf{F} = \langle -x, -y, z^3 \rangle$.

Solution: first, let's parametrize the surface

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = r, \qquad r \in [1,3], \theta \in [0, 2\pi].$$

With this parametrization we can compute

$$\mathbf{r}_\theta \times \mathbf{r}_r = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{bmatrix} \times \begin{bmatrix} \cos\theta \\ \sin\theta \\ 1 \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ -r \end{bmatrix}.$$

We want to confirm this is pointing in the direction of the downward normal. We know that the downward normal at the point $(2,0,2)$ should point in the direction $\langle 1, 0, -1 \rangle$ (verify by drawing!). Thus we want our cross product to be a positive scalar multiple of $\langle 1, 0, -1 \rangle$ and $(2,0,2)$. We observe that $\theta = 0, r = 2$ corresponds to the point $(2,0,2)$. Our cross product at this point is therefore $\langle 2, 0, -2 \rangle$, which is indeed a positive scalar multiple of $\langle 1, 0, -1 \rangle$. Thus this gives the desired orientation. Finally, we set up the integral

$$\int_0^{2\pi} \int_1^3 \mathbf{F}(\mathbf{r}) \cdot (\mathbf{r}_\theta \times \mathbf{r}_r)\, \mathrm{d}r\, \mathrm{d}\theta = \int_0^{2\pi} \int_1^3 \begin{bmatrix} -r\cos\theta \\ -r\sin\theta \\ r^3 \end{bmatrix} \cdot \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ -r \end{bmatrix} \mathrm{d}r\, \mathrm{d}\theta.$$