MATH 53 Quiz 5 (10/06) **Name:**

Please write legibly and explain your work clearly. Answers without explanations may receive less (or no) credit.

Problem 1. Consider the function

$$f(x, y) = xy$$

and the region where $x^2 + y^2 \leqslant 1$ and $y \leqslant 2x - 1$

(a) (6 points) Write out a complete step-by-step plan for how to find the maximum and minimum values of f on this domain by **using Lagrange multipliers**.

Solution:

1. Find critical points of f by solving $\nabla f = 0$.
2. Check which critical points from (1) satisfy the constraints $x^2+y^2 \leqslant 1$ and $y \leqslant 2x-1$. Save these for later.
3. Check the boundary $x^2 + y^2 = 1$ using Lagrange multipliers. This means define $g(x, y) = x^2 + y^2 - 1$ and solve the system of equations
$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}.$$
4. Check which critical points from (3) satisfy $y \leqslant 2x - 1$. Save these for later.
5. Check the boundary $y = 2x - 1$ using Lagrange multipliers. This means define $h(x, y) = y - 2x + 1$ and solve the system of equations
$$\begin{cases} \nabla f = \lambda \nabla h \\ h = 0 \end{cases}.$$
6. Check which critical points from (5) satisfy $x^2 + y^2 \leqslant 1$. Save these for later.
7. Find the points where both constraints are satisfied, i.e. solve $y = 2x - 1$ and $x^2 + y^2 = 1$.
8. Compare the value of f at the points from (2), (4), (6), and (7).

(b) (6 points) Execute your plan.

Solution:

1. $\nabla f = \langle y, x\rangle$, so $\nabla f = 0$ at $(0, 0)$.
2. We notice that $0^4 + 0^4 = 0 \leqslant 1$ but $0 - 2(0) + 1 = 1 \nleqslant 0$. We throw away $(0, 0)$.
3. $$\nabla f = \lambda \nabla g \rightsquigarrow \qquad y = \lambda 4x^3, \qquad x = \lambda 4y^3, \qquad x^4 + y^4 - 1 = 0.$$
   From the first two equations, we see that $x = \lambda^2 x \implies \lambda = \pm 1 \implies x = \pm y$. Plugging in to $x^2+y^2 = 1$ gets the solutions $x = \pm 1/\sqrt{2}$ and $y = \pm 1/\sqrt{2}$, which corresponds to 4 critical points: $(1/\sqrt{2}, 1/\sqrt{2}), (1/\sqrt{2}, -1/\sqrt{2}), (-1/\sqrt{2}, 1/\sqrt{2}), (-1/\sqrt{2}, -1/\sqrt{2})$.
4. Now we check which critical points satisfy $y - 2x + 1 \leqslant 0$.
   i. $(1/\sqrt{2}, 1/\sqrt{2})$: $1/\sqrt{2} - 2/\sqrt{2} + 1 = -1/\sqrt{2} + 1 > 1 - 1 = 0$, so we throw this point out.
   ii. $(-1/\sqrt{2}, 1/\sqrt{2})$: $\sqrt{2} + 2/\sqrt{2} + 1 > 1 > 0$, so we throw this point out.
   iii. $(1/\sqrt{2}, -1/\sqrt{2})$: $-\sqrt{2} - 2/\sqrt{2} + 1 = -3/\sqrt{2} + 1 < -3/2 + 1 = 1/2 \leqslant 0$, so we keep this point.
   iv. $(-1/\sqrt{2}, -1/\sqrt{2})$: $-\sqrt{2} + 2/\sqrt{2} + 1 = 1/\sqrt{2} + 1 > 1 > 0$, so we throw this point out.
5. $$\nabla f = \lambda \nabla h \rightsquigarrow \qquad y = -\lambda 2, \qquad x = \lambda, \qquad y - 2x + 1 = 0.$$
   The first two equations lead to $y = -2x$. We thus solve the equation $-2x - 2x + 1 = 0$ which gives us the critical point $(1/4, -1/2)$.
6. We compute $(1/4)^2 + (-1/2)^2 < 1/4 + 1/2 \leqslant 1$ so we save this point.
7. We solve $y = 2x - 1$ and $x^2 + y^2 = 1$, which yields $(4/5, 3/5)$ and $(0, -1)$.
8. Compare values.
   i. $f(1/\sqrt{2}, -1/\sqrt{2}) = -1/2$
   ii. $f(1/4, -1/2) = -1/8$
   iii. $f(0, -1) = 0$
   iv. $f(4/5, 3/5) = 12/25$

   From this we conclude that the minimum is at $(1/\sqrt{2}, -1/\sqrt{2})$ and the maximum is at $(4/5, 3/5)$, with the listed values.