MATH 53 Quiz 4 (09/22) **Name:**

Please write legibly and explain your work clearly. Answers without explanations may receive less (or no) credit.

Problem 1. (4 points)

(a) Let $\vec{\mathbf{u}} \in \mathbb{R}^2$ be a unit vector. Define in plain English the directional derivative in the direction $\vec{\mathbf{u}}$ of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at the point $\mathbf{p} \in \mathbb{R}^2$.

Solution: The rate at which f changes as we change the input from $\mathbf{p}$ in direction $\vec{\mathbf{u}}$.

(b) Now, let $f : \mathbb{R}^2 \to \mathbb{R}$ be a specific differentiable function such that

$$\nabla f(x, y) = \langle 3/2, \sqrt{3} \rangle.$$

For the following 3 questions, make sure your answer is a unit vector.

i. What is the direction $\vec{\mathbf{u}}$ that **maximizes** the directional derivative $D_{\vec{\mathbf{u}}} f$ at (x, y)?

Solution:

$$\left\langle \frac{3/2}{\sqrt{9/4+3}}, \frac{\sqrt{3}}{\sqrt{9/4+3}} \right\rangle$$

ii. What is the direction $\vec{\mathbf{u}}$ that **minimizes** the directional derivative $D_{\vec{\mathbf{u}}} f$ at (x, y)?

Solution:

$$\left\langle \frac{-3/2}{\sqrt{9/4+3}}, \frac{-\sqrt{3}}{\sqrt{9/4+3}} \right\rangle$$

iii. What is a direction $\vec{\mathbf{u}}$ that makes the directional derivative $D_{\vec{\mathbf{u}}} f$ at (x, y) **equal to 0**?

Solution:

$$\left\langle \frac{\sqrt{3}}{\sqrt{9/4+3}}, \frac{-3/2}{\sqrt{9/4+3}} \right\rangle$$

Problem 2. (4 points) What are all the critical points of $f(x,y) = (x^2+y^2)^2 - (x^2+y^2)$? *[Hint: to visualize the surface, try using polar coordinates.]*

Solution: We compute

$$\nabla f(x,y) = \left\langle 4x(x^2+y^2) - 2x, 4y(x^2+y^2) - 2y \right\rangle$$

And we try to solve $\nabla f = \langle 0, 0\rangle$. This leads to the equations

$$2x(2(x^2+y^2)-1) = 0 \qquad\qquad 2y(2(x^2+y^2)-1) = 0.$$

We observe that $x = 0$ and $x^2+y^2 = \frac{1}{2}$ both make the first equation zero.

1. If we take $x = 0$, then the second equation reads $2y(2y^2-1) = 0$ which has solutions at $y = \pm\frac{1}{\sqrt{2}}, 0$. The interesting critical point here is $(0,0)$.
2. When we consider the situation where $x^2+y^2 = \frac{1}{2}$, we notice that both equations are automatically zero. Thus, the circle of radius $\frac{1}{\sqrt{2}}$ are all critical points as well. This includes the points $\left(0, \pm\frac{1}{\sqrt{2}}\right)$ that we found earlier.

In conclusion, the critical points are $(0,0)$ and all the points on the circle of radius $1/\sqrt{2}$, i.e. the set $\{(x,y) : x^2+y^2 = 1/2\}$.

Problem 3. (4 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a twice-differentiable function, let $z = f(x,y)$, and let $x = y = r^2 s$. Compute

$$\frac{\partial^2 z}{\partial r \partial s}.$$

Solution: First we compute

$$\frac{\partial z}{\partial r} = f_x \frac{\partial x}{\partial r} + f_y \frac{\partial y}{\partial r} = 2rs(f_x(x,y) + f_y(x,y)).$$

Next we compute

$$\begin{aligned}
\frac{\partial}{\partial s}\left(\frac{\partial z}{\partial r}\right) &= \frac{\partial}{\partial s}(2rs(f_x+f_y)) = 2r(f_x+f_y) + 2rs\left(\frac{\partial f_x}{\partial s}(x,y) + \frac{\partial f_y}{\partial s}(x,y)\right) \\
&= 2r(f_x+f_y) + 2rs(r^2(f_{xx}+f_{xy}) + r^2(f_{xy}+f_{yy})) \\
&= 2r(f_x+f_y) + 2r^3 s(f_{xx} + 2f_{xy} + f_{yy})
\end{aligned}$$