MATH 54 Quiz 1 (09/01) **Name:**

Please write legibly and explain your work clearly. Answers without explanations may receive less (or no) credit.

Problem 1. (3 points) Compute the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

You may use that $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

Solution:

First, we parametrize the ellipse $x(t) = a\cos t$ and $y(t) = b\sin t$. Next, We compute $x'(t) = -a\sin t$. Now we know that we want to travel the entire ellipse clockwise, so we set up the area computation

$$A = \int_0^{\pi} y(t)x'(t)\,\mathrm{d}t = -\int_0^{2\pi} (-ab)\sin^2 t\,\mathrm{d}t = ab\int_0^{2\pi} \sin^2 t\,\mathrm{d}t = ab\int_0^{2\pi} \frac{1-\cos(2t)}{2}\,\mathrm{d}t = \pi ab$$

Problem 2. (3 points) Find the length of the curve $r = e^{2\theta}$ for $0 \leqslant \theta \leqslant \pi$.

Solution: We will use the formula for arc length of a polar curve

$$\int_0^{\pi} \sqrt{(r')^2 + r^2}\,\mathrm{d}\theta = \int_0^{\pi} \sqrt{(2e^{2\theta})^2 + (e^{2\theta})^2}\,\mathrm{d}\theta = \int_0^{\pi} e^{2\theta}\sqrt{5}\,\mathrm{d}\theta = \frac{\sqrt{5}}{2}e^{2\theta}\Big|_0^{\pi} = \frac{\sqrt{5}}{2}(e^{2\pi} - 1)$$

Problem 3. Consider the curve given by

$$x(t) = t^2 \qquad y(t) = t^3 + 3t^2.$$

(a) (1 point) Compute the slope at each point of the curve.

Solution:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 + 6t}{2t} = \frac{3(t+2)}{2}$$

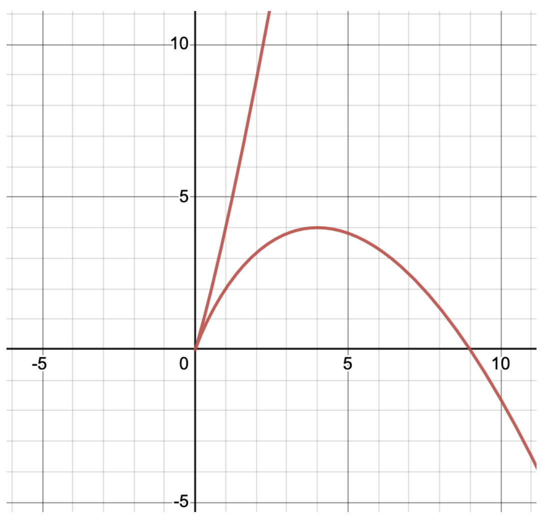
(b) (1 point) Compute the concavity at each point of the curve.

Solution

$$\frac{d^2y}{dx^2} = \frac{d(dy/dx)/dt}{dx/dt} = \frac{3/2}{2t} = \frac{3}{4t}$$

(c) (4 points) Plot the curve. Please label at least 4 important features.

Solution:



Possible features:

(a) $x > 0$.

(b) local max at $t = -2$ (what are the xy coordinates?)

(c) Concave up for $t > 0$, concave down for $t < 0$

(d) x-intercepts at $t = 0$ and $t = -3$ (what are the xy coordinates?)

(e) y-intercept at $t = 0$ (what are the xy coordinates?)

(f) cusp at $t = 0$, slope of the cusp

(g) direction of curve