Sample problems with explanations (preparation for midterm 1)

Students from Math 53 sections 108 and 111

Sep 25, 2023

I have categorized the work you all did for easier reference. You should be able to click on the headings in the table of contents and be taken to the appropriate section. Some people's work fit into multiple categories. In this case I categorized based on what I thought was the main theme.

Credit to each and every one of you to contributing to this document!

Contents

1 Parametric Curves (10.1-2)

1.1 [Pranav]

Monday, 11 September 2023 23:53

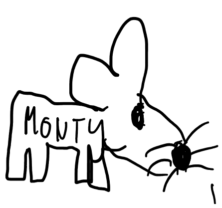
Catch the Mouse.

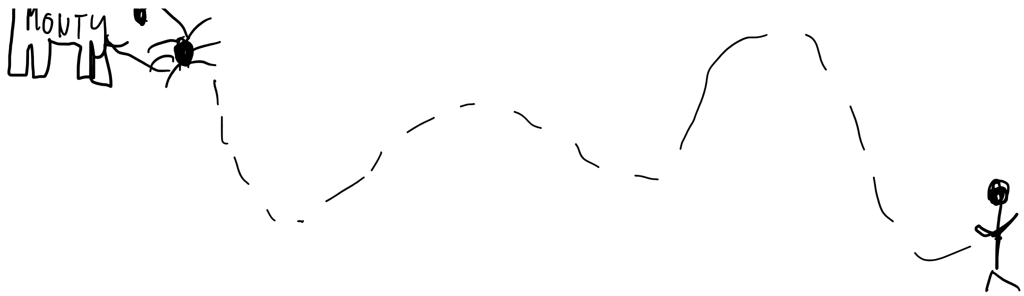
Monty the mouse has just executed a masterful heist and stolen the perfect slice of brie cheese you had been saving for the evening's dinner. However, unbeknowst to Monty you have become aware of his crime. Enraged you chase after monty as he swerves around and 10 seconds after he started his run you capture him Huzzah.

Whilst eating your reclaimed cheese you ponder the mighty mouse's path and realise that his displacement from the fridge during his run was

$$2\sin\frac{\pi t}{10}\,i + 2\cos\frac{\pi t}{10}\,j .$$

Find the length of the path Monty took before being caught as well as his acceleration and velocity at the instant before he was caught?





Solution: If we take our i position as our x coordinate and our j position as our y coordinate we get $x = 2\sin\frac{t}{10}$ and $y = 2\cos\frac{t}{16}$ as our parametric equations for our curve. We also know that I caught Monty in 10 seconds so $0 \leq t \leq 10$. As such using the length of a curve formula we have the length of the path L

$$L = \int_0^{10} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = \frac{2}{10}\cos\frac{t}{10} \qquad \frac{dy}{dt} = -\frac{2}{10}\sin\frac{t}{10}$$

$$L = \int_0^{10} \sqrt{\frac{4}{100}\cos^2\frac{t}{10} + \frac{4}{100}\sin^2\frac{t}{10}}\ dt$$

$$= \frac{1}{25} \int_0^{10} \sqrt{\cos^2 \frac{t}{10} + \sin^2 \frac{t}{10}} \, dt$$

$$= \frac{1}{25} \int_0^{10} 1 \, dt$$

$$= \frac{1}{25} \left[10 - 0 \right]$$

$$= \frac{10}{25} = \boxed{\frac{2}{5}}$$

so monty taveleled $\frac{2}{5}$ g a metre.

For part 2 we have the jollowing: If the displacement s is given by

$$s: \quad 2 \sin \frac{t}{10} \, i + 2 \cos \frac{t}{10} \, j$$

$$\frac{ds}{dt} \downarrow$$

$$v: \quad \frac{1}{5} \cos \frac{t}{10} \, i - \frac{1}{5} \sin \frac{t}{10} \, j$$

$$\frac{dv}{dt} \downarrow$$

$$a: \quad -\frac{1}{50} \sin \frac{t}{10} \, i - \frac{1}{50} \cos \frac{t}{10} \, j$$

at $t = 10$

$$\left\{ \begin{aligned} v &= \left(\tfrac{1}{5}\cos 1\right) i - \left(\tfrac{1}{5}\sin 1\right) j \\ a &= \left(-\tfrac{1}{50}\sin 1\right) i - \left(\tfrac{1}{50}\cos 1\right) j \end{aligned} \right\}$$

1.2 [Kamron]

Question: Find the length of the parametric equation given the formulas for x and y.

2) $x = \sin t, \quad y = -\cos t, \quad 0 \leq x \leq \pi$

1. Write out the equation to find the length of the parametric function.

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

2. Solve for the derivatives of the x and y equations.

$$\frac{dx}{dy} = \cos t$$

$$\frac{dy}{dt} = \sin t$$

3. Input these derivatives into the formula from Step 1.

$$= \int_0^{\pi} \sqrt{(\cos t)^2 + (\sin t)^2}\, dt$$

4. Simplify

$$= \int_0^{\pi} \sqrt{\cos^2 t + \sin^2 t} \; dt$$

5. Use the Pythagorean identity to further simplify.

$$= \int_0^{\pi} \sqrt{1} \; dt$$

6. Evaluate the integral and find the solution.

$$= [t]_0^{\pi}$$

$$= \pi - 0 = \pi$$

2 Polar Curves (10.3-4)

Uh-oh no one did a problem on this topic! Make sure to review it (maybe no one did it because no one remembered how it works 0.0)

3 Vectors and Regions in many Dimensions (12.1-2)

3.1 [Zachary]

Find the distance between two spheres defined to be

1. $x^2-16x+y^2+4y+z^2-8z=-75$
2. $(x-1)^2+(y-1)^2+(z+3)^2=7$

& then parameterize this line segment as a vector.

$x^2-16x+64+y^2+4y+4+z^2-8z+16=-75+64+4+16$

First, simplify the first equation to find the center of the first sphere

$(x-8)^2+(y+2)^2+(z-4)^2=9$

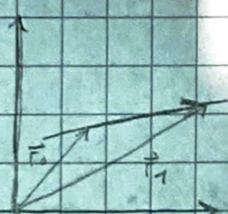
$d=\sqrt{(8-1)^2+(2-1)^2+(4+3)^2}$

$=\sqrt{99}$

$\sqrt{99}-\sqrt{7}-3$ ✓

The shortest distance between two spheres is on the line that connects the centers. & don't forget to subtract the radii

For the second part we will find a parameterized equation for the line segment connecting the centers.



$\vec{r}_0=\langle 1,1,3\rangle$

$\vec{r}_1=\langle 8,-2,4\rangle$

$\vec{r}_1-\vec{r}_0=\langle 7,-3,1\rangle$

$\vec{r}(t)=\langle 1,1,3\rangle+t\langle 7,-3,1\rangle$

$=\langle 1+7t,1-3t,3+t\rangle \quad 0\le t\le 1$

3.2 [Ashlee]

$P(-4,6,8)$, $Q(-3,1,3)$

shortest distance between a point and plane : distance between the point and the projection onto the plane.

① Find projection of P and Q on xy-plane : set z-coordinate to 0

projection of P : $(-4,6,0)$

projection of Q : $(-3,1,0)$

Only the z-coordinate varies, so the distance between the point and xy-plane is the absolute difference between the z-coord and plane $(z=0)$.

② P : $|8-0|=8$ > Q is closer to the xy-plane

Q : $|3-0|=3$

4 Dot and Cross Products (12.3-4)

4.1 [Dillon]

Problem #2

given vectors $\vec{a} <1, 3, 2>$ and $\vec{b} <3, 1, 4>$, find the volume of the pyramid with base $\vec{a}$, $\vec{b}$, $\vec{a}+\vec{b}$, and the origin and height of an orthogonal unit vector.

Step 1: Find orthogonal vector using cross product

$$<1,3,2> \times <3,1,4> = \begin{vmatrix} i & j & k \\ 1 & 3 & 2 \\ 3 & 1 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} i - \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} j + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} k$$

$$= (12-2)i - (4-6)j + (1-9)k$$

$$= 10i + 2j - 8k$$

$$= \boxed{<10, 2, -8>}$$

Step 2: convert orthogonal vector to unit vector

Step 2A: Find magnitude of orthogonal vector

$$M = \sqrt{10^2 + 2^2 + (-8)^2}$$

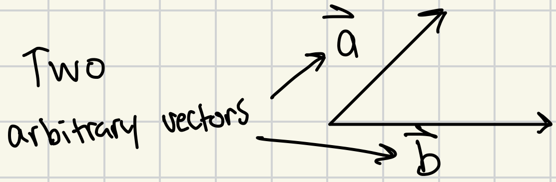
$$M = \sqrt{100 + 4 + 64}$$

$$M = \sqrt{168}$$

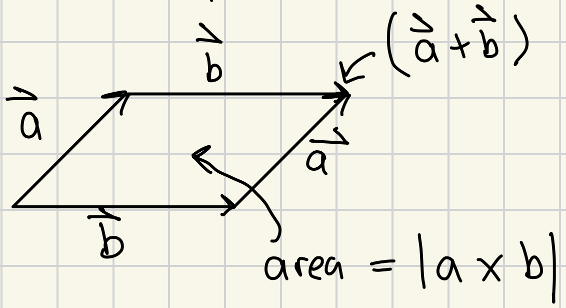
Step 2B: Divide original vector by magnitude to convert to unit vector

$$\frac{1}{\sqrt{168}} <10, 2, -8> = \vec{u}$$

Step3: Use definition of cross product to find area of base of pyramid



magnitude of cross product is equal to parallelogram formed by them



Note that if angle between $\vec{a}$ and $\vec{b}$ is zero, the area is zero this relates to the other definition of a cross product:

$$|\vec{a} \times \vec{b}| = |a||b|\sin\theta$$

if $\theta = 0$, then $\sin\theta = 0$, then $|\vec{a} \times \vec{b}| = 0$

From Step 2A, we know the magnitude is $\sqrt{168}$

Therefore, the area of the base is $\sqrt{168}$

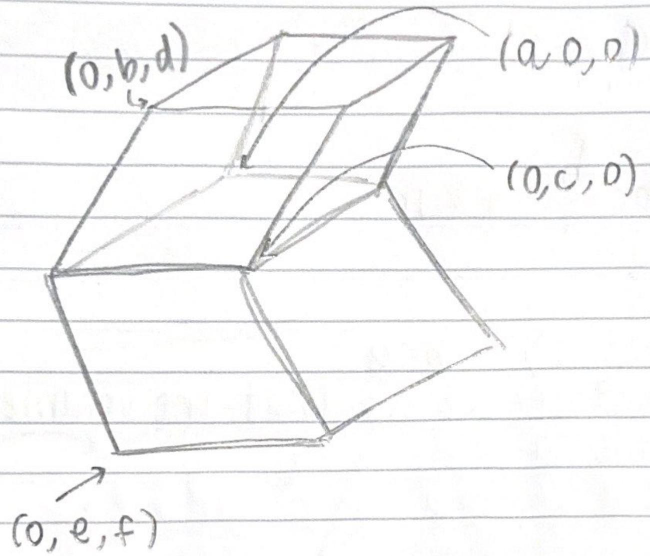
Step 4: Use formula $V = \frac{1}{3}Bh$ to find Volume

$|\vec{u}| = 1$ because it is a unit vector

So,

$$V = \frac{1}{3}\sqrt{168} \approx 4.32$$

4.2 [Aadhiti]



Find the volume of the prism to the left.

Solution:

We will start by finding the volume of the top prism.
We can represent this prism using the vectors
$\langle a,0,0\rangle, \langle 0,c,0\rangle, \langle 0,b,d\rangle$.
The volume of this prism can be denoted as
$V = |a \cdot (b \times c)|$
We will find the cross product of $\langle a,0,0\rangle$ and $\langle 0,c,0\rangle$.

$$\begin{vmatrix} i & j & k \\ a & 0 & 0 \\ 0 & c & 0 \end{vmatrix} = 0i + 0j + ack \rightarrow \langle 0,0,ac\rangle = \vec{v}_i$$

Then we will dot $\vec{v}_i$ with $\vec{a}$ as per the formula.
$\langle 0,0,ac\rangle \cdot \langle 0,b,d\rangle = acd$
Apply the same procedure to the bottom prism using the vectors $\langle 0,c,0\rangle, \langle a,0,0\rangle$ and $\langle 0,e,f\rangle$
The result will be $\langle 0,0,ac\rangle \cdot \langle 0,e,f\rangle = acf$

Thus, the final volume $acd + acf$, or $ac(d+f)$.

5 Lines and Planes (12.5)

5.1 [Daniel K]

1) Find an equation for the plane that passes through the points $(1,4,-2)$, $(3,5,6)$, and $(-2,1,3)$.

Solution:

1) Use the points to determine 2 vectors in the plane

$\vec{u} = (1,4,-2)-(3,5,6) = (-2,-1,-8)$

$\vec{v} = (1,4,-2)-(-2,1,3) = (3,3,-5)$

2) Take the cross product to find a normal vector

$\langle -2,-1,-8\rangle \times \langle 3,3,-5\rangle = \langle 29,-14,-3\rangle$

3) Use the normal vector and a point to write an equation for the plane

$29(x-1)-14(y-4)-3(z+2)=0$

5.2 [Jeremy]

Jeremy Mills | Midterm 1 Problem 1

Find the vector equation of the line that represents the intersection of the planes $2(x-1)+3(y-2)+4(z-3)=0$ and $3x+y-2z=0$.

Step 1. Find the normal vecters of the planes.

Using $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ equation, we know $\vec{n}=\langle a,b,c\rangle$ so...

$\vec{n}_1=\langle 2,3,4\rangle$ and $\vec{n}_2=\langle 3,1,-2\rangle$

Step 2. Find the cross product of the two normal vecters.

The cross product of two normal vecters gives the direction vecter of the intersection of the two planes.

$$\vec{n}_1\times\vec{n}_2=\langle -6-4,\ 12+4,\ 2-9\rangle$$
$$=\langle -10,16,-7\rangle$$

Step 3. Find a random ancher point on the intersection point.

Set the plane equations, and find a point:

$$2(\cancel{x}-1)+3(y-2)+4(z-3)=3x+y-2z$$
$$2x-2+3y-6+4z-12=3x+y-2z$$
$$-20=x-2y-6z$$

so a point to statisfy this is $(0,1,3)$

Step 4. Construct the equation.

$\vec{r}=\vec{r}_0+\vec{v}t$ $\quad\vec{v}=\langle -10,16,-7\rangle$

$\vec{r}=\langle 0,1,3\rangle+\langle -10,16,-7\rangle t$

$$\boxed{\vec{r}=\langle -10t,\ 1+16t,\ 3-7t\rangle}$$

Chapters Tested: 12.2, 12.4, 13.1

5.3 [Jeremy]

Jeremy Mills | Midterm 1 Problem 2

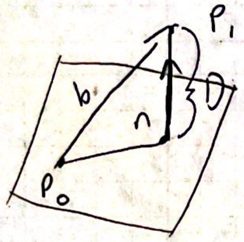
Find the distance between the point $(4,3,-2)$ and the plane $\vec{r}\cdot(7,-5,3)=2$

Step 1. Find the a, b, and c values of the equation of the plane.

since $\vec{r}\cdot(7,-5,3) = -7(x-x_0)-5(y-y_0)+3(z-z_0)-2$

$(a,b,c) = (7,-5,3)$ and $d=-2$

Step 2. Find the distance formula for a point and a plane.



$\vec{b} = \langle x-x_0,\ y-y_0,\ z-z_0\rangle$

$$D = |\text{proj}_n b| = \frac{|n\cdot b|}{|n|}$$

$$= \frac{|a(x-x_0)+b(y-y_0)+c(z-z_0)|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|ax+by+cz-(ax_0+by_0+cz_0)|}{\sqrt{a^2+b^2+c^2}},$$

but $ax_0+by_0+cz_0+d=0$

so

$$= \boxed{\frac{|ax+by+cz+d|}{\sqrt{a^2+b^2+c^2}}}$$

$\vec{n}$ is the normal vector,
P_0 is on the plane, P_1 is off the plane,
$\vec{b} = \overrightarrow{P_0P_1}$
D is the distance.

Step 3. Plug values into formula.

$a=7,\ b=-5,\ c=3,\ d=-2$

$x=4,\ y=3,\ z=-2$

$$D = \frac{|7(4)+3(-5)+3(-2)-2|}{\sqrt{7^2+(-5)^2+3^2}} = \boxed{\frac{5\sqrt{83}}{83}}$$

Chapters Tested: 12.2, 12.3, 12.5

5.4 [Daniel F]

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1. find an equation for the plane that passes through point $(-2,4,-3)$ & is perpendicular to planes

$-x+3y-5z=42$ $y-2z=5$

Solution: the normal vector to these two planes are $\langle -1,3,-5\rangle$ & $\langle 0,1,-2\rangle$

to find a vector perpendicular to $-x+3y-5z=42$ & $y-2z=5$, you must take the cross product of their normal vectors

$\langle -1,3,-5\rangle \times \langle 0,1,-2\rangle$

$\langle (3\cdot(-2) - (-5)\cdot 1), -(-1\cdot(-2) - (-5)\cdot 0, -1\cdot 1 - 3\cdot 0 \rangle$

$= \langle -1,-2,-1\rangle$

vector that is perpendicular to $-x+3y-5z=42$ & $y-2z=-5$

to compute plane that passes through point $(-2,4,-3)$ we have to use $\langle -1,-2,-1\rangle$ dotted by a vector which

contains the difference in x/y/z & the point $(-2,4,-3)$

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5.5 [Jacob]

Consider the function $f(x,y) = x^3+3xy^2-2y^3$ in the xy-plane. You are standing at the point $(2,1)$ and want to walk in direction V such that the directional derivative of f at the point is maximized.

$$\nabla f(2,1) = \langle 3x^2+3y^2,\ 6xy-6y^2 \rangle$$
$$= \langle 15, 6 \rangle$$
$$\nabla f \cdot \vec{v} = |\nabla f||\vec{v}| \cos\theta \qquad \cos\theta = 1 \quad |\vec{v}| = 1$$
$$\langle 15, 6 \rangle \cdot \vec{v} = \sqrt{225+36}$$
$$\vec{v} = \frac{\nabla f}{|\nabla f|} \qquad \vec{v} = \frac{\langle 15,6 \rangle}{\sqrt{261}}$$
$$\vec{v} = \left\langle \frac{15}{\sqrt{261}}, \frac{6}{\sqrt{261}} \right\rangle$$

Explanation:

First we must the gradient vector at the point (2,1)

Next we must determine at what unit vector will that directional derivative be maximized, since we know that the directional derivative is the gradient vector dot a unit vector, we can also determine that the dot product of these two vectors is equal to the product of the magnitude of these two vectors and cosine. Since we are trying to maximize cosine must be equal to one because that is the greatest cosine can be.

Therefore we can determine that the direction which will maximize the directional derivative is when the gradient vector points in the same direction as the unit vector. Thus we turn the gradient vector into the unit vector, which is the direction that will maximize our directional derivative.

5.6 [Rahul]

*→ Midterm Study Bonus

1. What is the equation of the plane that intersects the line $x = 1 + 3t,\ y = 7 - 2t,\ z = 5 - t$ @ $t=0$ and is parallel to the plane

$11x + 4y + 2z = 2$?

Relevant concepts:

- The parametric equations for a line through the pt. (x_0, y_0, z_0) and parallel to the direction vector $\langle a, b, c\rangle$ are:
  $x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$

- The scalar equations of the plane through point $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c\rangle$ is
  $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, or
  $Ax + By + Cz = D$

Solution:

- If we let $t = 0$, a possible point in the plane is $(1, 7, 5)$

- The normal vector of the parallel plane is $\langle 11, 4, 2\rangle$ from the coefficients A, B, and C.

- substituting the pt $(1,7,5)$ for (x_0, y_0, z_0) and $\langle 11, 4, 2\rangle$ for $\langle a, b, c\rangle$ into the equation of a plane:

$$11(x - 1) + 4(y - 7) + 2(z - 5) = 0$$

$$\Rightarrow 11x - 11 + 4y - 28 + 2z - 10 = 0$$

$$\Rightarrow \boxed{11x + 4y + 2z = 49}$$

- Section 12.5

5.7 [Vansh]

Determine the equation for a plane that goes through the point (1, -2, 3) and is at right angles to the planes represented by the equations 2x + y - 3z = 12 and 4x - 2y + z = 7.

Point = $(1, -2, 3)$ Planes → $2x+y-3z=12$
$4x-2y+z=7$

Step 1: Find Cross product to obtain the normal to planes as the third plane is orthogonal to both.

$$\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} = \begin{pmatrix} 1-6 \\ -12-2 \\ -4-4 \end{pmatrix}$$

$$= \begin{pmatrix} -5 \\ -14 \\ -8 \end{pmatrix}$$

step 2:
Now, we can obtain the eq of the new plane using the normal and point given with formula:

$n = \langle a, b, c \rangle$, $P_0(x_0, y_0, z_0)$

↓

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

Hence, equation of plane is

$$-5(x-1) - 14(y+2) - 8(z-3) = 0$$

$$-5x + 5 - 14y - 28 - 8z + 24 = 0$$

$$-5x - 14y - 8z + 1 = 0$$

$$\boxed{5x - 14y - 8z = 1}$$

Is the eq of the plane that is orthogonal to $2x + y - 3z = 12$ and $4x - 2y + z$ and also passes through $(1, -2, 3)$.

6 Calculus of Parametrics (13.1-2)

6.1 [Ashlee]

$r(t) = \langle t^3+5, t^2, 10-t \rangle$

Use unit tangent vector formula: $T(t) = \dfrac{r'(t)}{|r'(t)|}$

① Find derivative of vector function, $r'(t)$.

$r'(t) = \langle 3t^2, 2t, -1 \rangle$

② Find length of derivative of vector function, $|r'(t)|$.

$|r'(t)| = \sqrt{(3t^2)^2+(2t)^2+(-1)^2} = \sqrt{9t^4+4t^2+1}$

③ Assemble to find unit tangent vector, $T(t)$.

$$T(t) = \left\langle \frac{3t^2}{\sqrt{9t^4+4t^2+1}}, \frac{2t}{\sqrt{9t^4+4t^2+1}}, \frac{-1}{\sqrt{9t^4+4t^2+1}} \right\rangle$$

6.2 [Nishan]

Let $r(t) = \langle x(t), y(t), z(t) \rangle$ represent an objects position at time t, where $x(t) = -t^2 + 4t$ $y(t) = \frac{1}{60}(40t)^{3/2}$ $z(t) = e^{\pi} + \sqrt{48}\,t$

a) find $r'(t)$

Sol: $r'(t) = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k = (-2t+4) i + (\sqrt{40t}) j + (\sqrt{48}) k$

Compute derivatives using power and chain rule

b) find the distance of the path of the object from time $t_0 = 2$ to $t = 5$ [the distance of the path is just the arc length in 3d]

Sol: $L = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$

$$L = \int_2^5 \sqrt{(-2t+4)^2 + (\sqrt{40t})^2 + (\sqrt{48})^2}\, dt$$

$$= \int_2^5 \sqrt{4t^2 - 16t + 16 + 40t + 48}\, dt$$

$$= \int_2^5 \sqrt{4(t^2 - 8t + 16)}\, dt$$ [factoring out a 4 makes quadratic easier to factor]

$$= 2\int_2^5 \sqrt{(t-4)^2}\, dt = 2\int_2^5 t-4\, dt = \tfrac{1}{2}t^2 - 4t \Big|_2^5$$

[integral power rule]

$$= \boxed{2\left(\left[\frac{25}{2} - 20\right] - [2 - 8]\right)}$$

7 Multivariable Functions and Level Sets (14.1)

Uh-oh no one did a problem on this topic! Make sure to review it (maybe no one did it because no one remembered how it works 0.0)

8 Limits, Continuity, Differentiability (14.2)

8.1 [Mustafa]

2) Show whether or not the function $f(x,y) = x^3y - xy + y^3 + 5$ is differentiable at the point $(0,1)$.

① The theorem "if the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b)" indicate that we can first find the partial derivatives of f.

$f_x(x,y) = 3x^2y - y$ $\quad$ $f_x(0,1) = 3(0)(1) - 1 = -1$

$f_y(x,y) = x^3 - x + 3y^2$ $\quad$ $f_y(0,1) = 0 - 0 + 3(1^2) = 3$

Both f_x and f_y can exist for any values of x and y.

To prove continuity at a point, the following must be true:

- The limit of the function as it approaches that point must exist.
- The function must exist at that point.
- The value of the function at that point and the limit at that point must be equal.

$$\lim_{(x,y)\to(0,1)} f_x(x,y) = \lim_{(x,y)\to(0,1)} (3x^2y - y) = \lim_{(x,y)\to(0,1)} (0 - 1) = -1 = f_x(0,1)$$

$$\lim_{(x,y)\to(0,1)} f_y(x,y) = \lim_{(x,y)\to(0,1)} (x^3 - x + 3y^2) = \lim_{(x,y)\to(0,1)} (0 - 0 + 3) = 3 = f_y(0,1)$$

Both f_x and f_y meet the conditions to be continuous at $(0,1)$ $\therefore$ $f(x,y) = x^3y - xy + y^3 + 5$ is differentiable at $(0,1)$.

Intuitively, since f_x and f_y are defined for all x and y, and since they are not piecewise functions, we can assume they are continuous.

8.2 [Kaycee]

Question 1: What is the directional derivative of $f(x,y) = 3x^3y + \frac{4}{5}y^2$ in the direction $u = \langle 0,3 \rangle$? In words, explain where f increases the fastest. Given P(1,0), what is the maximum rate of change of f?

Solution 1:

The equation for a directional derivative is $D_u f(x,y) = \nabla f(x,y) \cdot u$, with $\nabla f(x,y) = \langle f_x, f_y \rangle = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$

Step 1: Find $\frac{\partial f}{\partial x}$:

$f(x,y) = 3x^3y + \frac{4}{5}y^2$

$f_x = \frac{d}{dx}3x^3 \cdot y + 3x^3 \cdot \frac{d}{dx}y + \frac{d}{dx}4y^2$ → product rule

$f_x = 9x^2 \cdot y + 3x^3 \cdot 0 + 0$ → since we are changing in x, keeping y constant, $\frac{d}{dx}y = 0$

$f_x = 9x^2y$

Step 2: find $\frac{df}{dy}$:

$f_y = \frac{d}{dy}3x^3 \cdot y + 3x^3 \cdot \frac{d}{dy}y + \frac{d}{dy}4y^2$ → product rule

$f_y = 0 \cdot y + 3x^3 \cdot 1 + 8y$ → since we are changing in y, keeping x constant, $\frac{d}{dx}x = 0$

$f_y = 3x^3 + 8y$

Step 3: Combine f_x, f_y, and u in the directional derivative equation.

$D_u f(x,y) = \nabla f(x,y) \cdot u$,

$= \langle f_x, f_y \rangle \cdot \langle u \rangle$

$= \langle 9x^2y, 3x^3 + 8y \rangle \langle 0,3 \rangle$

$\boxed{D_u f(x,y) = 9x^3 + 24y}$

Step 4: Explain where f increases the fastest.

Since the gradient vector points in the direction of greatest increase, f will increase the fastest in the direction of the gradient vector.

Step 5: Find the maximum rate of change

Since going in the direction of the gradient vector will increase f the fastest, the maximum rate of change is given by:

$|\nabla f(1,0)| = |\langle 9(1)^2(0), 3(1)^3 + 8(0) \rangle|$

$= |\langle 0,3 \rangle|$ → finding the magnitude of

$= (0)^2 + (3)^2$ → the vector.

$\boxed{|\nabla f(1,0)| = 9}$

The maximum rate of change of f @ P(1,0) is 9.

9 Partial Derivatives and Chain Rule (14.3,5)

9.1 [Daniel K]

2) Let $z=f(x,y)$, where $x=s^2t$ and $y=s^3+t^3$. Find $\frac{\partial^2 z}{\partial s \partial t}$.

1) $\frac{\partial^2 z}{\partial s \partial t}=\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial s}\right)=\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\right)$ ← Partial derivative chain rule

2) Solve for inner function: $\frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}=f_x(2st)+f_y(3s^2)$

3) Take derivative with respect to t, remembering product rule

$\frac{\partial}{\partial t}\left(f_x(2st)+f_y(3s^2)\right)=2sf_x+2st\left(\frac{\partial f_x}{\partial t}\right)+3s^2\left(\frac{\partial f_y}{\partial t}\right)$

4) Apply chain rule again

$2sf_x+2st\left(\frac{\partial f_x}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f_x}{\partial y}\frac{\partial y}{\partial t}\right)+3s^2\left(\frac{\partial f_y}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f_y}{\partial y}\frac{\partial y}{\partial t}\right)$

$=2sf_x+2st\left(f_{xx}(s^2)+f_{xy}(3t^2)\right)+3s^2\left(f_{yx}(s^2)+f_{yy}(3t^2)\right)$

$=2sf_x+2s^3tf_{xx}+6st^3f_{xy}+3s^4f_{xy}+9s^2t^2f_{yy}$

9.2 [Dillon]

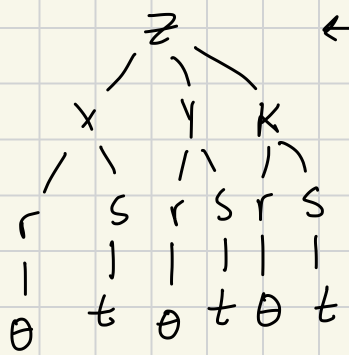
find $\frac{d^2 z}{d\theta\, dt}$ if $z = f(x, y, k)$ and $r = 2\sin\theta$

$x = x(r, s)$ $\quad$ $s = \cos(2t)$

$y = y(r, s)$

$k = k(r, s)$

Step 1: Write out tree diagram of relationships:



← z is function of x, y, k

← x, y, k are functions of r and s

← r is a function of θ, s is a function of t

Step 2: Find $\frac{dz}{d\theta}$

Note: Due to equality of mixed partials, could find $\frac{dz}{dt}$ first, up to preference.

step 2A: Identify all paths for $\frac{dz}{d\theta}$ using tree diagram

all ways a change in θ affects z

Step 2B: Begin to apply chain rule:

$$\frac{dz}{d\theta} = \underbrace{\frac{dz}{dx}\frac{dx}{dr}\frac{dr}{d\theta}}_{\text{path 1}} + \underbrace{\frac{dz}{dy}\frac{dy}{dr}\frac{dr}{d\theta}}_{\text{path 2}} + \underbrace{\frac{dz}{dk}\frac{dk}{dr}\frac{dr}{d\theta}}_{\text{path 3}}$$

Step 2C: Find $\frac{dr}{d\theta}$ to simplify

$$r = 2\sin\theta$$
$$\frac{dr}{d\theta} = 2\cos\theta$$

step 2D: substitute $\frac{dr}{d\theta}$

$$\frac{dz}{d\theta} = \frac{dz}{dx}\frac{dx}{dr}(2\cos\theta) + \frac{dz}{dy}\frac{dy}{dr}(2\cos\theta) + \frac{dz}{dK}\frac{dk}{dr}(2\cos\theta)$$

step 3: Evaluate Second derivative

$$\frac{d}{dt}\left[\frac{dz}{d\theta}\right] = \underbrace{\frac{d}{dt}\left[\frac{dz}{dx}\frac{dx}{dr}(2\cos\theta)\right]}_{\text{Part A}} + \underbrace{\frac{d}{dt}\left[\frac{dz}{dy}\frac{dy}{dr}(2\cos\theta)\right]}_{\text{Part B}} + \underbrace{\frac{d}{dt}\left[\frac{dz}{dK}\frac{dk}{dr}(2\cos\theta)\right]}_{\text{Part C}}$$

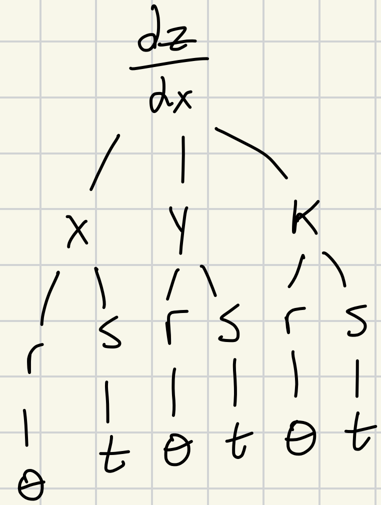
step 3A: Evaluate Part A:

$2\cos\theta$ does not depend on t. Treat as constant, move out.

$$2\cos\theta \, \frac{d}{dt}\left[\frac{dz}{dx}\frac{dx}{dr}\right]$$

Apply product rule:

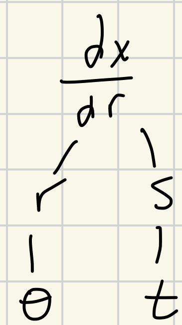
$$2\cos\theta\left[\frac{d}{dt}\frac{dz}{dx}\frac{dx}{dr} + \frac{dz}{dx}\frac{d}{dt}\frac{dx}{dr}\right]$$



Use chain rule to evaluate $\frac{d}{dt}\frac{dz}{dx}$

$$\frac{d}{dt}\frac{dz}{dx} = \frac{d^2z}{dx^2}\frac{dx}{ds}\frac{ds}{dt} + \frac{d^2z}{dxdy}\frac{dy}{ds}\frac{ds}{dt} + \frac{d^2k}{dxdk}\frac{dk}{ds}\frac{ds}{dt}$$

use Chain rule to evaluate $\frac{d}{dt}\frac{dx}{dr}$



$$\frac{d}{dt}\frac{dx}{dr} = \frac{d^2x}{drds}\frac{ds}{dt}$$

Substitute:

$$\boxed{2\cos\theta\left[\frac{dx}{dr}\left(\frac{d^2z}{dx^2}\frac{dx}{ds}\frac{ds}{dt} + \frac{d^2z}{dxdy}\frac{dy}{ds}\frac{ds}{dt} + \frac{d^2k}{dxdk}\frac{dk}{ds}\frac{ds}{dt}\right) + \frac{dz}{dx}\left(\frac{d^2x}{drds}\frac{ds}{dt}\right)\right]}$$

<u>Step 3B</u>: Evaluate Part B

$$\frac{d}{dt}\left[\frac{dz}{dy}\frac{dy}{dr}(2\cos\theta)\right]$$

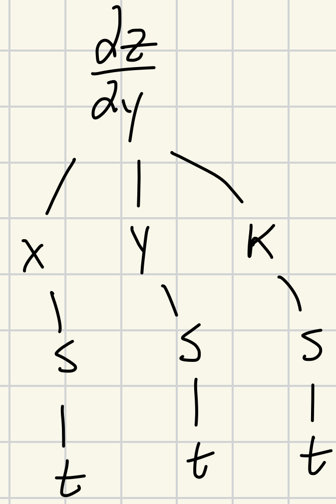
$$= 2\cos\theta\,\frac{d}{dt}\left[\frac{dz}{dy}\frac{dy}{dr}\right]$$

$$= 2\cos\theta\left[\frac{d}{dt}\frac{dz}{dy}\frac{dy}{dr} + \frac{dz}{dy}\frac{d}{dt}\frac{dy}{dr}\right]$$

Use chain rule to Evaluate $\frac{d}{dt}\frac{dz}{dy}$

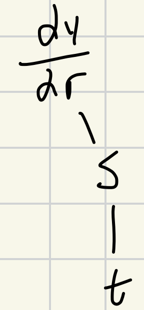
$$\frac{\partial}{\partial t}\frac{\partial z}{\partial y} = \frac{\partial}{\partial x}\frac{\partial z}{\partial y}\frac{dx}{ds}\frac{\partial s}{\partial t} + \frac{\partial}{\partial y}\frac{\partial z}{\partial y}\frac{dy}{ds}\frac{\partial s}{\partial t} + \frac{\partial}{\partial k}\frac{\partial z}{\partial y}\frac{dk}{ds}\frac{\partial s}{\partial t}$$

$$= \frac{\partial^2 z}{\partial x\partial y}\frac{dx}{ds}\frac{\partial s}{\partial t} + \frac{\partial^2 z}{\partial y^2}\frac{dy}{ds}\frac{\partial s}{\partial t} + \frac{\partial^2 z}{\partial k\partial y}\frac{dk}{dS}\frac{\partial s}{\partial t}$$



use Chain rule to evaluate $\frac{\partial}{\partial t}\frac{\partial y}{\partial r}$

$$\frac{\partial}{\partial t}\frac{\partial y}{\partial r} = \frac{\partial}{\partial s}\frac{\partial y}{\partial r}\frac{\partial s}{\partial t} = \frac{\partial^2 y}{\partial s\partial r}\frac{\partial s}{\partial t}$$



Substitute:

$$\boxed{2\cos\theta\left[\frac{\partial y}{\partial r}\left(\frac{\partial^2 z}{\partial x\partial y}\frac{dx}{ds}\frac{\partial s}{\partial t} + \frac{\partial^2 z}{\partial y^2}\frac{dy}{ds}\frac{\partial s}{\partial t} + \frac{\partial^2 z}{\partial k\partial y}\frac{dk}{dS}\frac{\partial s}{\partial t}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial^2 y}{\partial s\partial r}\frac{\partial s}{\partial t}\right)\right]}$$

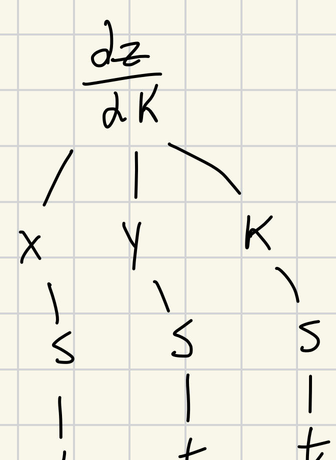
Part 3B

$$\frac{\partial}{\partial t}\left[\frac{\partial z}{\partial k}\frac{\partial k}{\partial r}(2\cos\theta)\right]$$

$$= 2\cos\theta\,\frac{\partial}{\partial t}\left[\frac{\partial z}{\partial k}\frac{\partial k}{\partial r}\right]$$

$$= 2\cos\theta\left[\frac{\partial}{\partial t}\frac{\partial z}{\partial k}\frac{\partial k}{\partial r} + \frac{\partial z}{\partial k}\frac{\partial}{\partial t}\frac{\partial k}{\partial r}\right] \leftarrow \text{product rule}$$

Evaluate $\frac{\partial}{\partial t}\frac{\partial z}{\partial k}$



$$\frac{\partial}{\partial t}\frac{\partial z}{\partial k} = \frac{\partial}{\partial x}\frac{\partial z}{\partial k}\frac{dx}{ds}\frac{\partial s}{\partial t} + \frac{\partial}{\partial y}\frac{\partial z}{\partial k}\frac{dy}{ds}\frac{\partial s}{\partial t} + \frac{\partial}{\partial z}\frac{\partial z}{\partial k}\frac{dk}{ds}\frac{\partial s}{\partial t}$$

$$\frac{\partial}{\partial t}\frac{\partial z}{\partial k} = \frac{\partial^2 z}{\partial x\partial k}\frac{dx}{dS}\frac{\partial s}{\partial t} + \frac{\partial^2 z}{\partial y\partial k}\frac{dy}{dS}\frac{\partial s}{\partial t} + \frac{\partial^2 z}{\partial z\partial k}\frac{dk}{dS}\frac{\partial s}{\partial t}$$

Evaluate $\frac{d}{dt}\frac{dk}{dr}$

$$\frac{dk}{dr}$$
$$\quad \backslash$$
$$\quad s$$
$$\quad |$$
$$\quad t$$

$$= \frac{d}{ds}\frac{dk}{dr}\frac{ds}{dt}$$

$$= \frac{d^2k}{dsdr}\frac{ds}{dt}$$

Substitute:

$$\boxed{2\cos\theta\left[\frac{dk}{dr}\left(\frac{d^2z}{dxdk}\frac{dx}{ds}\frac{ds}{dt}+\frac{d^2z}{dydk}\frac{dy}{ds}\frac{ds}{dt}+\frac{d^2z}{dzdk}\frac{dk}{ds}\frac{ds}{dt}\right)+\frac{dz}{dk}\left(\frac{d^2k}{dsdr}\frac{ds}{dt}\right)\right]}$$

<u>Part 4</u>

Find $\frac{ds}{dt}$

$$s = \cos(2t)$$

$$\frac{ds}{dt} = -2\sin(2t)$$

substitute into Part A, B, C.
I'm not going to do all the substitution and simplification because my hand hurts from writing & its just tedious. When simplifying, mixed partials are equivilent.

9.3 [Keshav]

Midterm Study

Bonus

1. Question: The Cobb-Douglas production function relates economic performance (P) to labor (L) and capital invested (K).

Textbook page 929: $P = 1.01\left(\frac{L}{2}\right) \cdot K^2$

The exponential terms a and b vary with time according to the following functions:

$L = e^{0.2t}$ $K = \ln(1+t)$ t is measured in years

Using these equations, find how a country's economic performance varies with time if t = 4 years, P = 105; K = 130

Solution: Want to find $\frac{dP}{dt}$

Use chain rule: $\frac{dP}{dt} = \frac{\partial P}{\partial L}\frac{\partial L}{\partial t} + \frac{\partial P}{\partial K}\frac{\partial K}{\partial t}$

$\frac{\partial P}{\partial L} = \frac{1.01 \cdot K^2}{2}$ $\frac{\partial P}{\partial K} = 1.01LK$ Find partial derivatives

$\frac{\partial L}{\partial t} = 0.2e^{0.2t}$ $\frac{\partial K}{\partial t} = \frac{1}{1+t}$

Plug in values

$\left.\frac{dP}{dt}\right|_{t=4, L=105, K=130} = \left(\frac{1.01 \cdot K^2}{2}\right) 0.2e^{0.2t} + \frac{1.01LK}{1+t}$

$= \frac{1.01}{2} \cdot (130)^2 \cdot 0.2e^{0.2(4)} + \frac{1.01(105)(130)}{1+4} = \boxed{6556.08}$

9.4 [Nishan]

z is a function f(x, y) which has continuous third order partial derivatives, find $\frac{\partial z}{\partial r}$, $\frac{\partial^2 z}{\partial r^2}$, $\frac{\partial^3 z}{\partial r^3}$

if $x = 3tr^2$ and $y = 4r$

Solutions

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}6tr + \frac{\partial z}{\partial y}(4)$$

[Since z is a function of x, y we have to take partial derivatives with respect to x and y].

$$\frac{\partial^2 z}{\partial r^2} = 6t\frac{\partial z}{\partial x} + 6tr\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 4\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$$

$\frac{\partial}{\partial r}\left[\frac{\partial z}{\partial x}\right]$ let $q = \frac{\partial z}{\partial x} \Rightarrow \frac{\partial}{\partial r}[q] = \frac{\partial q}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial q}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial q}{\partial x}6t + \frac{\partial q}{\partial y}(4)$

[for finding these partials, treat $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ as any variable]

$\frac{\partial}{\partial r}\left[\frac{\partial z}{\partial y}\right]$ let $v = \frac{\partial z}{\partial y} \Rightarrow \frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}6t + \frac{\partial v}{\partial y}4$

$$\frac{\partial^2 z}{\partial r^2} = 6t\frac{\partial z}{\partial x} + 6tr\left[\frac{\partial^2 z}{\partial x^2}6t + \frac{\partial^2 z}{\partial x \partial y}4\right] + 4\left[\frac{\partial^2 z}{\partial y \partial x}6t + \frac{\partial^2 z}{\partial y^2}4\right]$$

Note, $\frac{\partial q}{\partial x} = \frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial x} = \frac{\partial^2 z}{\partial x^2}$, applies to $\frac{\partial q}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

$$= 6t\frac{\partial z}{\partial x} + 36t^2r\frac{\partial^2 z}{\partial x^2} + 24tr\frac{\partial^2 z}{\partial x \partial y} + 24t\frac{\partial^2 z}{\partial y \partial x} + 16\frac{\partial^2 z}{\partial y^2}$$

instead of adding new variables, f_{xx} and f_{xy} will be used

$$\frac{\partial^3 z}{\partial r^3} = 6t\frac{\partial}{\partial r}(f_x) + \underbrace{36t^2\frac{\partial^2 z}{\partial x^2} + 36t^2r\frac{\partial f_{xx}}{\partial r}}_{\text{product rule}} + 24t\frac{\partial^2 z}{\partial x \partial y} + 24tr\frac{\partial}{\partial r}(f_{xy})$$
$$+ 24t\frac{\partial}{\partial r}[f_{xy}] + 16\frac{\partial}{\partial r}(f_{yy})$$
$$= 6t\left[\frac{\partial^2 z}{\partial x^2}6t + \frac{\partial^2 z}{\partial x \partial y}4\right] + 36t^2\frac{\partial^2 z}{\partial x^2} + 36t^2r\left[\frac{\partial f_{xx}}{\partial x}6tr + \frac{\partial f_{xx}}{\partial y}4\right] + 24t\frac{\partial^2 z}{\partial x \partial y}$$
$$+ 24tr\left[\frac{\partial f_{xy}}{\partial x}6tr + \frac{\partial f_{xy}}{\partial y}4\right] + 24t\left[\frac{\partial f_{xy}}{\partial x}6tr + \frac{\partial f_{xy}}{\partial y}4\right] + 16\left[\frac{\partial f_{yy}}{\partial x}6tr + \frac{\partial f_{yy}}{\partial y}4\right]$$

Color coded highlights correspond to expanded partial derivatives

9.5 [Rahul]

2. If $z = 4\cos(x^2) + e^{2y} + 417$, and

$x = 4st + e^s - u$; $y = 3\sqrt{s} + t^2 + \sin(u)$

find $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$, $\frac{\partial z}{\partial u}$, leave in terms of x, y, s, t, and u.

Relevant Concepts:

- To find f_x, regard y as a constant and differentiate $f(x,y)$ w/ respect to x
- To find f_y, regard x as a constant and differentiate $f(x,y)$ w/ respect to y

- Chain rule (general):

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for $i = 1, 2, \ldots, m$

Solution:

$$\frac{\partial z}{\partial s} = \overset{①}{\frac{\partial z}{\partial x}}\overset{②}{\frac{\partial x}{\partial s}} + \overset{③}{\frac{\partial z}{\partial y}}\overset{④}{\frac{\partial y}{\partial s}}$$

$$= \underbrace{-2x(4)\sin(x^2)}_{①} \cdot \underbrace{(4t + e^s)}_{②} + \underbrace{2e^{2y}}_{③} \cdot \underbrace{\left(\tfrac{3}{2}s^{-\frac{1}{2}}\right)}_{④}$$

$$= \boxed{-8x\sin(x^2)(4t+e^s) + \frac{3e^{2y}}{\sqrt{s}}}$$

$$\frac{\partial z}{\partial t} = \overset{①}{\frac{\partial z}{\partial x}}\overset{②}{\frac{\partial x}{\partial t}} + \overset{③}{\frac{\partial z}{\partial y}}\overset{④}{\frac{\partial y}{\partial t}}$$

$$= \underbrace{-8x\sin(x^2)}_{①} \cdot \underbrace{(4s)}_{②} + \underbrace{2e^{2y}}_{③} \cdot (2t)$$

$$= \boxed{-32x\sin(x^2)s + 4e^{2y}t}$$

$$\frac{\partial z}{\partial u} = \overset{①}{\frac{\partial z}{\partial x}}\overset{②}{\frac{\partial x}{\partial u}} + \overset{③}{\frac{\partial z}{\partial y}}\overset{④}{\frac{\partial y}{\partial u}}$$

$$= \underbrace{-8x\sin(x^2)}_{①} \cdot \underbrace{(-1)}_{②} + \underbrace{2e^{2y}}_{③} \cdot (\cos(u))$$

$$= \boxed{8x\sin(x^2) + 2e^{2y}\cos(u)}$$

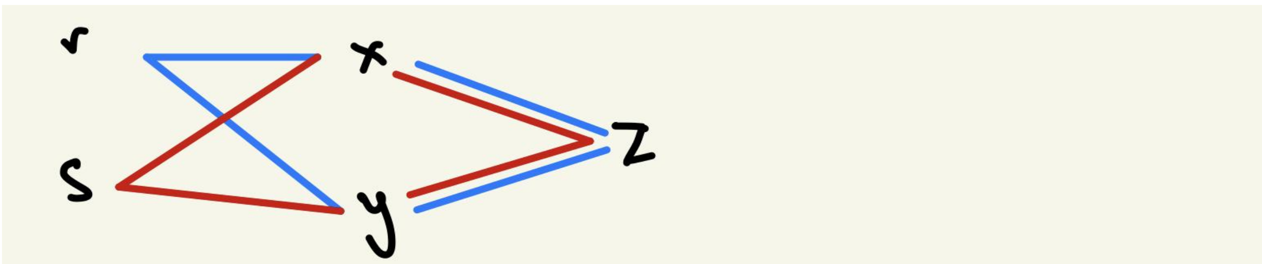
- Sections 14.3, 14.5

9.6 [Kamron]

Question: Find the partial derivative of z with regard to r and s. Use the chain rule to determine the equation.

$$1)\ z = x^2 + x^2y^2 + y^2, \quad x = r^2s^2 + s, \quad y = r^2s^2 + r$$

1. Draw a box diagram showing all the potential “paths” to find the partial derivative of z with regard to r and s.



2. Begin by choosing between r or s. We will start with r then move to s. Write out an equation depicting the partial derivatives which will comprise the final equation for the partial derivative of z with regard to r.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{dy}{\partial r}$$

3. Find the partial derivative of z with respect to x and y.

$$\frac{\partial z}{\partial x} = 2x + 2xy^2 \qquad \frac{\partial z}{\partial y} = 2yx^2 + 2y$$

4. Find the partial derivative of x and y with respect to r.

$$\frac{\partial x}{\partial r} = 2rs^2 \qquad \frac{dy}{\partial r} = 2rs^2 + 1$$

5. Rewrite the equation from step 2 using the derivatives we found.

$$= (2x + 2xy^2)(2rs^2) + (2yx^2 + 2y)(2rs^2 + 1)$$

6. Simplify

$$= (4rs^2x + 4rs^2xy^2) + (4rs^2x^2y + 2x^2y + 4rs^2y + 2y)$$

$$= 4rs^2x + 4rs^2xy^2 + 4rs^2x^2y + 2x^2y + 4rs^2y + 2y$$

7. Now we solve with the variable s. Write out an equation depicting the partial derivatives which will comprise the final equation for the partial derivative of z with regard to s.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}$$

8. Find the partial derivative of x and y with respect to s.

$$\frac{\partial x}{ds} = 2sr^2 + 1 \qquad \frac{\partial y}{\partial s} = 2sr^2$$

9. Rewrite the equation from step 7 using the derivatives we found in step 3 and 8.

$$= (2x + 2xy^2)(2sr^2 + 1) + (2yx^2 + 2y)(2sr^2)$$

10. Simplify

$$= (4r^2sx + 2x + 4r^2sxy^2 + 2xy^2)(4r^2sx^2y + 4r^2sy)$$

$$= 4r2sx + 2x + 4r^2sxy^2 + 2xy^2 + 4r^2sx^2y + 4r^2sy$$

9.7 [Aadhiti

P2 For $f(x,y) = 15x^7y^{10} + 10x\sqrt{y} + 12y$, prove that Clairut's Theorem holds.

P2. Clairut's theorem states f_{xy} is equal to f_{yx}.

$f(x,y) = 15x^7y^{10} + 10x\sqrt{y} + 12y$

we will rewrite this as $f(x,y) = 15x^7y^{10} + 10xy^{1/2} + 12y$.

$f_x(x,y) = 105x^6y^{10} + 10y^{1/2}$ $\quad f_y = 150x^7y^9 + 5xy^{-1/2} + 12$

$f_{xy} = 1050x^6y^9 + 5y^{-1/2}$ $\quad f_{yx} = 1050x^6y^9 + 5y^{-1/2}$

$f_{xy} = f_{yx}$

9.8 [Vansh]

Consider a differentiable function, Z(u, v) = (u^3) + (u^2)v where u and v are related by the equations u = (p^3)q and v = (p^2)+ (1/q). Use the chain rule to express the partial derivatives ∂Z/∂p and ∂Z/∂q when p = 1 and q = 2.

$Z = u^3 + u^2 v$ $\quad u = p^3 q \ , \ v = p^2 + \frac{1}{q}.$

Need to find $\partial z/\partial p$ and $\partial z/\partial q$ when $p = 1$ and $q = 2$.

Step 1: set up chain rule expressions

$$\frac{\partial z}{\partial p} = \frac{\partial z}{\partial u} \times \frac{\partial u}{\partial p} + \frac{\partial z}{\partial v} \times \frac{\partial v}{\partial p}$$

$$\frac{\partial z}{\partial q} = \frac{\partial z}{\partial u} \times \frac{\partial u}{\partial q} + \frac{\partial z}{\partial v} \times \frac{\partial v}{\partial q}$$

Step 2: Evaluate the partial derivatives

$$\frac{\partial z}{\partial u} = 3u^2 + 2uV \qquad \frac{\partial z}{\partial v} = u^2$$

Made with Goodnotes

$$\frac{\partial u}{\partial p} = 3qp^2 \qquad \frac{\partial u}{\partial q} = p^3$$

$$\frac{\partial v}{\partial p} = 2p \qquad \frac{\partial v}{\partial q} = \frac{-1}{q^2}$$

Step 3: Find final expressions

$$\frac{\partial z}{\partial p} = (3u^2 + 2uv)(3qp^2) + (u^2)(2p)$$

$$\frac{\partial z}{\partial q} = (3u^2 + 2uv)(p^3) + (u^2)\left(\frac{-1}{q^2}\right)$$

Step 4: Evaluate possible u and v values

$u = p^3 q$ so when $p = 1$ $q = 2$

$u = (1)^3(2) = 2$

$$V = p^2 + \frac{1}{q} = (1)^2 + \frac{1}{2}$$

$$= \frac{3}{2}$$

step 5: Finally use these values to find the final answer

$$\frac{\partial z}{\partial p} = \left(3(2)^2 + 2(2)\left(\frac{3}{2}\right)\right)\left(3(2)(1)^2\right) + (2)^2\left(2(1)\right)$$

$$\frac{\partial z}{\partial q} = \left(3(2)^2 + 2(2)\left(\frac{3}{2}\right)\right)\left((1)^3 + (2)^2\left(\frac{-1}{(2)^2}\right)\right)$$

$$\frac{\partial z}{\partial p} = \left(12 + 6\right)(6) + (4)(2)$$

$$= (18)(6) + 8 = 116$$

Made with Goodnotes

$$\frac{\partial z}{\partial q} = (18)(1) + (h)\left(\frac{-1}{h}\right)$$

$$= 18 - 1 = 17$$

10 Tangent Planes, Directional Derivatives and Gradient (14.4,6)

10.1 [Zachary]

14.6

Where on the surface of the shape defined by the equation

$x^2 + 2y^2 + 3z^2 = 9$

are the tangent planes parallel to the plane

$x + y + z = 1$? (no need to find plane)

$F(x,y,z) = x^2 + 2y^2 + 3z^2$

$\nabla F = \langle 2x, 4y, 6z \rangle$ normal vector

$\langle 2x, 4y, 6z \rangle = k \langle 1, 1, 1 \rangle$

$2x = k \quad 4y = k \quad 6z = k$

$x = k/2 \quad y = k/4 \quad z = k/6$

$(k/2)^2 + 2(k/4)^2 + 3(k/6)^2 = 9$

$k^2/4 + 2(k^2/16) + 3(k^2/36) = 9$

$k^2/4 + k^2/8 + k^2/12 = 9$

$\frac{2k^2}{8} + k^2/8 + k^2/12 = 9$

$\frac{9k^2}{24} + \frac{2k^2}{24} = 9$

$11k^2/24 = 9$

$k = \pm\sqrt{\frac{24 \cdot 9}{11}} = \pm\sqrt{\frac{216}{11}}$

$\langle 2x_0, 4y_0, 6z_0 \rangle = \sqrt{\frac{216}{11}} \langle 1, 1, 1 \rangle$

$x_0 = \pm 1/2 \sqrt{\frac{216}{11}} \quad y_0 = \pm 1/4 \sqrt{\frac{216}{11}}$

$z_0 = \pm 1/6 \sqrt{\frac{216}{11}}$

Here it is useful to conceptualize the surface as a level surface of a function of three variables where the output is 9

Then this becomes the gradient

The tangent plane is parallel when the normal vectors are parallel

Finding a point on the ~~[illegible]~~ surface where this is true

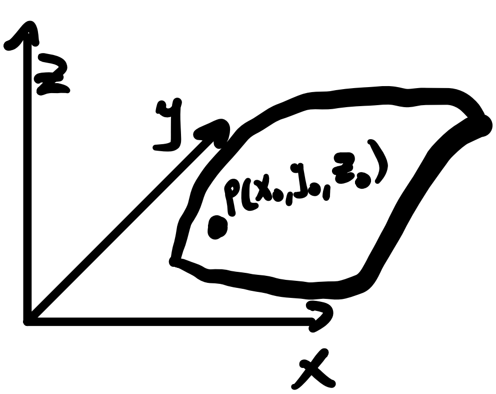
So these points define plane

10.2 [Alvaro]

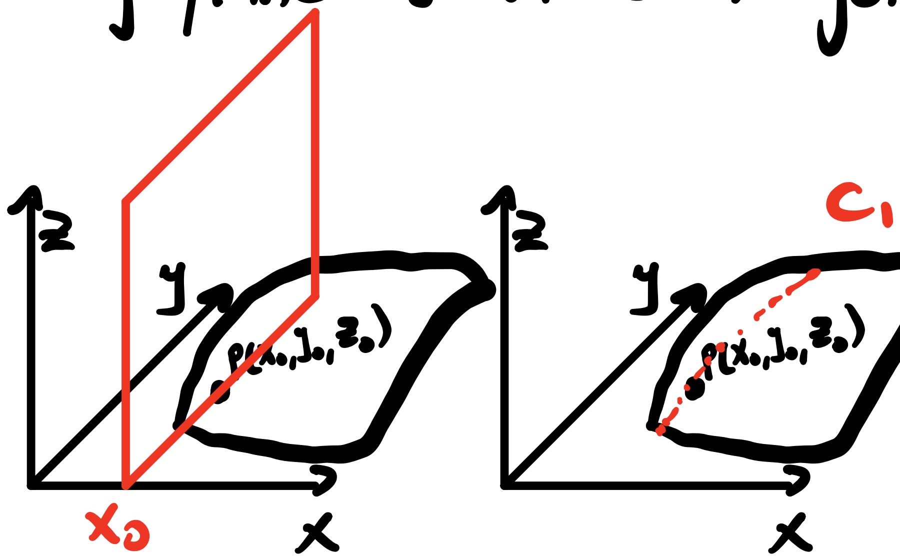
Tangent Planes

Given a surface S, $S \in \mathbb{R}^3$
$\iff z = f(x, y)$, $\mathbb{R}^2 \to \mathbb{R}^1$
$\wedge \; P(x_0, y_0, z_0) \in S$
$\wedge \; C_1 \equiv x \cap S, x = x_0 \wedge C_2 \equiv y \cap S, y = y_0$
$\implies P(x_0, y_0, z_0) \in C_1 \wedge P(x_0, y_0, z) \in C_2$

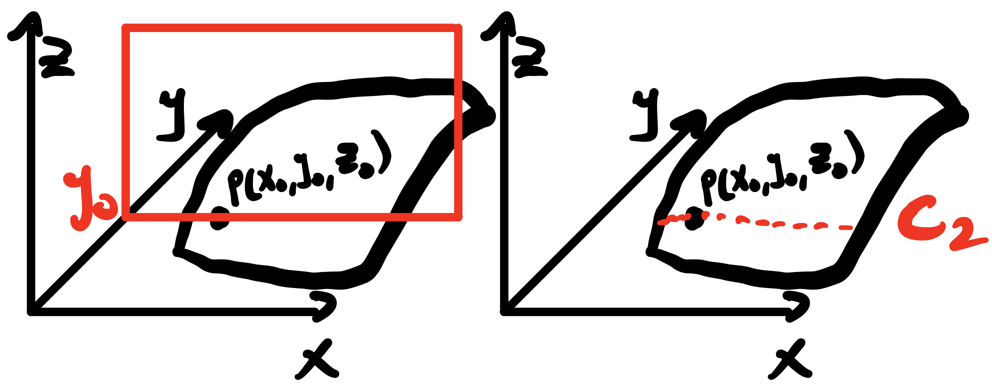
Given a surface S with equation $z = f(x, y)$ (maps inputs x and y to output z) which exists in 3 dimensions (1 dimension in output space, and 2 dimensions in input space) and there exists a point $P(x_0, y_0, z_0)$, the point P lies on the curve C_1 as a result on the intersection between the plane $x = x_0$, and lies on the curve C_2 as a result on the intersection between the plane $y = y_0$.



Intersecting plane x_0 with S to get C_1

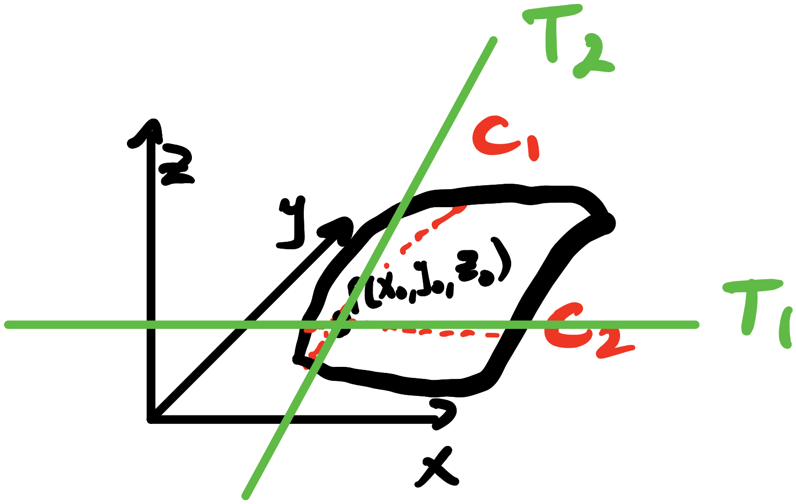


Intersecting plane y_0 with S to get C_2



$$\Longleftrightarrow T_1 \text{ ⟂̸ } C_1 \quad (\text{⟂̸} = \text{tangent})$$
$$\wedge\ T_2 \text{ ⟂̸ } C_2$$
$$\Rightarrow \{T_1, T_2\} \in T,\ T \text{ ⟂̸ } S$$

If T_1 is the tangent line to the curve C_1 and T_2 is the tangent line to the curve C_2, then the tangent plane T at point P contains both T_1 and T_2.



If any plane R passes through point $p(x_0, y_0, z_0)$, then

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$
$$\frac{A}{C}(x-x_0)+\frac{B}{C}(y-y_0)+z-z_0=0$$
$$-\frac{A}{C}(x-x_0)-\frac{B}{C}(y-y_0)-z+z_0=0$$
$$-\frac{A}{C}(x-x_0)-\frac{B}{C}(y-y_0)=z-z_0$$

$$\iff a=-\frac{A}{C} \wedge b=-\frac{B}{C}$$
$$\Rightarrow -\frac{A}{C}(x-x_0)-\frac{B}{C}(y-y_0)=a(x-x_0)+b(y-y_0)$$
$$\Rightarrow a(x-x_0)+b(y-y_0)=z-z_0$$
$$T \equiv a(x-x_0)+b(y-y_0)=z-z_0$$
$$\iff T \cap X,\ x=x_0$$
$$\Rightarrow T_1 \equiv a(x-x)+b(y-y_0)=z-z_0$$
$$a(0)+b(y-y_0)=z-z_0$$
$$T_1 \equiv b(y-y_0)=z-z_0,\ b=f_y(x_0,y_0)$$

$\Leftrightarrow T \cap y,\ y = y_0$

$$\Rightarrow T_2 \equiv a(x - x_0) + b(y - y) = z - z_0$$

$$a(x - x_0) + b(0) = z - z_0$$

$$T_2 \equiv a(x - x_0) = z - z_0,\ a = f_x(x_0, y_0)$$

If any plane that passes through the point $P(x_0, y_0, z_0)$ can be described with the equation $a(x - x_0) + b(y - y_0) = z - z_0$ then the tangent plane can also be described with the same equation. If the previous equation represents the tangent plane, then the intersection between the tangent plane and the plane $x = x_0$ must be the tangent line T_1. Similarly the intersection between the tangent plane and the plane $y = y_0$ must be the tangent line T_2.

Find the tangent plane at point $(100,0)$ at surface $z = \cos(x\cdot y)\cdot y$

$$\iff f(x,y) = \cos(x\cdot y)\cdot y$$

$$\Rightarrow f_x(x,y) = \frac{\partial}{\partial x}\left[\cos(x\cdot y)\cdot y\right]$$

$$= y\,\frac{\partial}{\partial x}\left[\cos(x\cdot y)\right]$$

$$= -y\sin xy\cdot\frac{\partial}{\partial x}\left[x\cdot y\right]$$

$$= -y^2\sin(xy)$$

$$f_x(100,0) = -(0)^2\sin((100)(0))$$

$$= 0$$

$$\Rightarrow f_y(x,y) = \frac{\partial}{\partial y}\left[\cos(x\cdot y)\cdot y\right]$$

$$= \frac{\partial}{\partial y}\left[\cos(x\cdot y)\right]\cdot y + \cos(x\cdot y)\cdot\frac{\partial}{\partial y}\left[y\right]$$

$$= -y\sin(xy)\cdot\frac{\partial}{\partial y}(x\cdot y) + \cos(x\cdot y)$$

$$= -yx\sin(xy) + \cos(x\cdot y)$$

$$f_y(100,0) = -(0)(100)\sin((100)(0)) + \cos((100)\cdot(0))$$

$$= 1$$

$$\iff z - z_0 = a(x-x_0) + b(y-y_0)$$

$$\Rightarrow T \doteq z - z_0 = (0)(x-x_0) + (1)(y-y_0),\ a=0,\ b=1$$

$$z - z_0 = y - y_0$$

$$\iff z_0 = f(100,0) = 100 \wedge y_0 = 0$$

$$\Rightarrow \underline{z = y + 100}$$

10.3 [Alvaro]

Directional Derivatives and Gradient

Given a surface S with equation $z = f(x, y)$. If we want to find the rate of change of z at a point $P(x_0, y_0)$ in the direction of an arbitrary unit vector $\vec{u} = \langle a, b \rangle$, we need to consider that the plane that passes through point P in the direction of $\vec{u}$ intersects S in a curve C. the slope of the tangent line of the curve as a result of the intersection of the plane that passes through point P in the direction of $\vec{u}$ is the rate of change of z in the direction of $\vec{u}$.

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

where $\vec{u} = \langle a, b \rangle$

$$D_{\vec{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

where $\vec{u} = \langle a, b \rangle$

The Gradient Vector

$\Leftrightarrow D_{\vec{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b$

$\wedge\ \vec{u} \in \mathbb{R}^2,\ \vec{u} = \langle a, b \rangle$

$\wedge\ \vec{v} \in \mathbb{R}^2,\ \vec{v} = \langle f_x(x,y), f_y(x,y) \rangle$

$\Rightarrow \vec{v} \cdot \vec{u} = f_x(x,y)a + f_y(x,y)b$

$\Rightarrow D_{\vec{u}} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \cdot \vec{u}$

$\langle f_x(x,y), f_y(x,y) \rangle = \vec{\nabla} f$

$\Rightarrow D_{\vec{u}} f(x,y) = \vec{\nabla} f \cdot \vec{u}$

Find the directional derivative and gradient in surface $z = e^{x-y} + xy$ at point 0,0 in directions $\theta = \frac{\pi}{2}$, $\theta = \frac{\pi}{4}$ and $\theta = 0$

$$:\iff D_{\vec{u}} f(x,y) = \vec{\nabla} f \cdot \vec{u}$$

$$\wedge \ \vec{\nabla} f = \langle f_x(x,y), f_y(x,y) \rangle$$

$$\wedge \ \vec{u} = \langle a, b \rangle$$

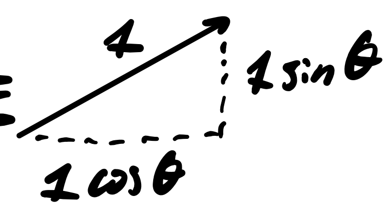
$$\iff f(x,y) = e^{x-y} + xy$$

$$\implies f_x(x,y) = e^{x-y} + x$$

$$\wedge \ f_y(x,y) = -e^{x-y} + y$$

$$\underline{\vec{\nabla} f = \langle e^{x-y} + x, -e^{x-y} + y \rangle}$$

$$\iff \vec{u} = \langle a, b \rangle, \forall \vec{u} \equiv$$



$$\wedge \ \theta = \frac{\pi}{2}$$

$$\implies \vec{u}_1 = \langle \cos\frac{\pi}{2}, \sin\frac{\pi}{2} \rangle, \ \theta = \frac{\pi}{2}$$

$$= \langle 0, 1 \rangle$$

$$\vec{u}_2 = \langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4} \rangle, \ \theta = \frac{\pi}{6}$$

$$= \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$$

$$\vec{u}_3 = \langle \cos 0, \sin 0 \rangle, \quad \theta = 0$$
$$= \langle 1, 0 \rangle$$

$$D_{\vec{u}} f(x,y) = \vec{\nabla} f \cdot \vec{u}_1, \quad \theta = \frac{\pi}{2}$$
$$= \langle e^{x-y}+x, -e^{x-y}+y \rangle \cdot \langle 0, 1 \rangle$$
$$= -e^{x-y}+y$$
$$D_{\vec{u}} f(0,0) = -e^{(0)-(0)}+(0) = \underline{-1}$$

$$D_{\vec{u}} f(x,y) = \vec{\nabla} f \cdot \vec{u}_2, \quad \theta = \frac{\pi}{6}$$
$$= \langle e^{x-y}+x, -e^{x-y}+y \rangle \cdot \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$$
$$D_{\vec{u}} f(0,0) = \frac{\sqrt{3}}{2}\left(e^{(0)-(0)}+(0)\right) + \frac{1}{2}\left(-e^{(0)-(0)}+(0)\right)$$
$$= \frac{\sqrt{3}}{2} - \frac{1}{2} = \underline{\frac{\sqrt{3}-1}{2}}$$

$$D_{\vec{u}} f(x,y) = \vec{\nabla} f \cdot \vec{u}_3, \quad \theta = 0$$
$$= \langle e^{x-y}+x, -e^{x-y}+y \rangle \cdot \langle 1, 0 \rangle$$
$$= (e^{x-y}+x) + 0(-e^{x-y}+y)$$
$$D_{\vec{u}} f(0,0) = e^{(0)-(0)}+(0) = \underline{1}$$

10.4 [Elaine]

Suppose that over a certain region of space the electrical potential V is given by $V(x,y,z) = 2x^2 + yz + xyz$. (a) Find the rate of change of the potential at $P(1,1,-1)$ in the direction of the vector $v = \langle 3,4,5 \rangle$.

SOLUTION:

$$\nabla V = \langle 4x + yz,\ z + xz,\ y + xy \rangle$$

find $\langle V_x, V_y, V_z \rangle$ which is the gradient of f.

plug in $(1,1,-1)$

$$\nabla V(1,1,-1) = \langle 4 + (1)(-1),\ (-1) + (1)(-1),\ (1) + (1)(1) \rangle$$

$$\langle 3, -2, 2 \rangle$$

$$D_v = \frac{v}{|v|} \cdot \nabla V = \frac{\langle 3,4,5 \rangle}{\sqrt{3^2+4^2+5^2}} = \left\langle \frac{3}{\sqrt{50}}, \frac{4}{\sqrt{50}}, \frac{5}{\sqrt{50}} \right\rangle \cdot \langle 3,-2,2 \rangle$$

$$\sqrt{9+16+25}$$

Solve for directional derivative using the formula and $v = \langle 3,4,5 \rangle$

do the dot product to solve!

$$= \frac{9}{\sqrt{50}} + \frac{-8}{\sqrt{50}} + \frac{10}{\sqrt{50}} = \boxed{\frac{11}{\sqrt{50}}}$$

remember to find the distance of $v \rightarrow \sqrt{x_0^2 + y^2 + z^2}$

10.5 [Elaine]

Dimensions of a rectangular ~~toy~~ are measured to be 80 cm, 60 cm, 40 cm & each measurement is correct to within 0.3 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

volume of box $= xyz$ so

$$dV = \frac{dV}{dx}dx + \frac{dV}{dy}dy + \frac{dV}{dz}dz$$

$$= yz\,dx + xz\,dy + xy\,dz$$

> Since $|\Delta x| \leq 0.3$, $|\Delta y| \leq 0.3$ & $|z| \leq 0.3$.

> Estimation of largest error in the volume is made by making $dx = 0.3$, $dy = 0.3$ & $dz = 0.3$ & $x = 80$ $y = 60$ $z = 40$.

← calculating the error of the total volume which is about dV using the values above.

$$\Delta V \approx dV = (60)(40)(0.3) + (80)(40)(0.3) + (80)(60)(0.3)$$

$$= 720 + 960 + 1440 = \boxed{3120}$$

Thus an error of 0.3 cm in measuring each dimension could lead to an error of approximately 3120 cm^3 in the calculated volume!

↖ solution!!

10.6 [Mustafa]

1. Let $l = f(x,y,z) = 8x^3 + 9y^2 - \frac{4}{z}$. Find an equation for the tangent plane to $f(x,y,z)$ at the input $(\frac{1}{2}, 3, 1)$ by using the level surface of a function.

Solution:

① set a function $g = l - f(x,y,z) = l - \left(8x^3 + 9y^2 - \frac{4}{z}\right) = 0$

(subtracting one value from an equivalent value yields 0). This allows us to use the 0 level surface of g to find the tangent plane.

② $f(\frac{1}{2}, 3, 1) = 8\left(\frac{1}{2}\right)^3 + 9(3)^2 - \frac{4}{1} = 1 + 81 - 4 = 78$

At the point $(\frac{1}{2}, 3, 1)$ $l = 78$

③ Find ∇g by taking the partial derivatives of g with respect to each input.

$\nabla g = \langle g_x, g_y, g_z, g_l \rangle = \langle -24x^2, -18y, -\frac{4}{z^2}, 1 \rangle$

④ Plug in the given values for x, y, z

$\nabla g(\frac{1}{2}, 3, 1, 78) = \langle -24\left(\frac{1}{2}\right)^2, -18(3), -\frac{4}{1^2}, 1 \rangle$

$\nabla g(\frac{1}{2}, 3, 1, 78) = \langle -6, -54, -4, 1 \rangle$

Gradient vectors at a point along a level surface are always orthogonal to the tangent through that point on the level surface.

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(5) The equation for a tangent plane to a level surface can be written in this case as

$$g_x(\tfrac{1}{2}, 3, 1, 78)(x - \tfrac{1}{2}) + g_y(\tfrac{1}{2}, 3, 1, 78)(y - 3) + g_z(\tfrac{1}{2}, 3, 1, 78)(z - 1) + g_l(\tfrac{1}{2}, 3, 1, 78)(l - 78) = 0$$

x_0, y_0, z_0, l_0

$$\begin{array}{r} 169 \\ -\ 78 \\ \hline 91 \end{array}$$

$$0 = -6(x - \tfrac{1}{2}) + (-54)(y - 3) + (-4)(z - 1) + 1(l - 78)$$

$$0 = -6x + 3 - 54y + 162 - 4z + 4 + l - 78$$

$$-l = -6x - 54y - 4z + 3 + 162 + 4 - 78$$

$$\boxed{l = 6x + 54y + 4z - 91}$$

move l to right side and divide by -1

Additional notes: The equation for a tangent plane to a level surface is the dot product between the gradient vector and $\langle x - x_0, y - y_0, z - z_0 \rangle$ (tangent at a point on the level surface). The gradient vector is orthogonal to this, so their cross product will always be 0 ($\cos 90° = 0$).

10.7 [Jacob]

Find an equation for the plane that passes through the point (x_0, y_0, z_0) and is perpendicular to the planes $5z = 2x + 10y - 20$ and $y = x + 5$

$20 = 2x + 10y - 5z \qquad -5 = x - y$

Normal vectors: $\langle 2, 10, -5 \rangle$ & $\langle 1, -1, 0 \rangle$

$\langle 2, 10, -5 \rangle \times \langle 1, -1, 0 \rangle = \langle -5, -5, -12 \rangle$

$-5(x - x_0) - 5(y - y_0) - 12(z - z_0) = 0$

Explanation:

First we must find the normal vectors so we can cross them and use the normal vector produced by those two vectors to write the equation of the plane which is perpendicular to the two planes and passes through the desired point.

We can rewrite it in the standard form of a plane equation (Ax + By + Cz = D).

Now we take the coefficients of x, y, and z to get the normal vector for both of the planes.

We can find the normal vector for the plane we want (perpendicular to both of them). The normal vector of our desired plane will be the cross product of other planes normal vectors

Now, we can use this normal vector and the point to write the equation of the plane.

10.8 [Kaycee]

Question 2: Find the limit: $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^2+y^2}$

Solution 2:

To approach this problem, we want to check if the limit exists by approaching from different directions to see if they equal.

Step 1: Check if the limit exists at $x = 0$

$\lim_{(0,y)\to(0,0)} \frac{(0)^2y^2}{2(0)^2+y^2} = \frac{0}{y^2} = 0$ $\longrightarrow$ notice that we changed $\lim_{(x,y)} \longrightarrow \lim_{(0,y)}$

Step 2: Check if the limit exists at $y = 0$

$\lim_{(x,0)\to(0,0)} \frac{x^2(0)}{2x^2+(0)^2} = 0$

Step 3: Check if the limit exists at $y = x$

$\lim_{(x,x)\to(0,0)} \frac{x^4}{2x^2+x^2} = 0$

Since approaching f from these directions are equal, they do not tell us whether or not a limit exists. Since we still need to approach f from multiple directions, we can switch to polar coordinates to confirm the limit equals to 0. We can use polar coordinates because it allows us to approach a point from any direction whenever we have two variables in our function. Therefore, we can use the following to solve:

$x = r\cos\theta$
$y = r\sin\theta$
$r^2 = x^2+y^2$

Step 4: substitute in polar coordinates

$\lim_{r\to 0} \frac{(r\cos\theta)^2(r\sin\theta)}{2(x^2+y^2)}$ $\rightarrow$ notice: $\lim_{(x,y)\to(0,0)} \rightarrow \lim_{r\to 0}$

$= \lim_{r\to 0} \frac{(r\cos\theta)^2(r\sin\theta)}{2r^2}$

$= \lim_{r\to 0} \frac{r^2\cos^2\theta \cdot r\sin\theta}{2r^2}$

$= \lim_{r\to 0} \frac{\cos^2\theta \cdot r\sin\theta}{2}$

$= \frac{\cos^2\theta \cdot (0)\sin\theta}{2}$

$= \frac{0}{2}$

$= \boxed{0}$

Since the $\lim_{r\to 0} \frac{(r\cos\theta)^2(r\sin\theta)}{2(x^2+y^2)} = 0$, we can confirm that

$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^2+y^2} = 0$

11 Maxima and Minima (14.7)

11.1 [Keshav]

⑦ Question: Take a function f that has second partial derivatives

$f_{xx} = 3x^2y + 2ax$

$f_{xy} = 2y/x$ ← constant

$f_{yy} = 2y^2$

Assuming that f has a critical point at $f(1,-1)$, find the domain of values constant a needs to take for f to have a saddle point there.

Solution: Use $D = f_{xx} \cdot f_{yy} - [f_{xy}]^2$

If $D < 0$, the function has a saddle point

$D = (3x^2y - 2ax) \cdot 2y^2 - [\frac{2y}{x}]^2 < 0$ ← Plug in 2nd derivatives

$D = (3(1)(-1) - 2a(1)) \cdot 2 - [-2]^2 < 0$ ← Plug in $(x,y) = (1,-1)$

$-4a - 6 - 4 < 0$ $\quad \boxed{a > -\frac{5}{2}}$ ← Solve equation

11.2 [Daniel F]

for the function $f(x,y) = 2x^3 + y^2 - 6xy + 4y$

Find its min/max & saddle point

$f_x = 6x^2 - 6y$ $\quad f_y = 2y - 6x + 4$

Solve for 0

$6x^2 - 6y = 0$ $\quad 2y - 6x + 4 = 0$

$x^2 = y$ $\quad y = 3x - 2$

$x^2 = 3x - 2$

$x^2 = y$

$2^2 = 4$

$1^2 = 1$

$x^2 - 3x + 2 = 0$

$(x-2)(x-1)$

$x = 2, 1$

$y = 4, 1$

Critical points are $P(1,1)$ & $Q(2,4)$

now we used 2nd deriv test to determine.

$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

$f_{xx} = 12x$

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