Math 53 Midterm 2 Practice

1 Lagrange Multipliers

1. [Ryan] Use Lagrange multipliers to find the maximum volume of a rectangular prism with fixed surface area.
2. [Elliot and Ryan] Given the function $f(x, y) = x^2 + y^2$ subject to the constraint $x^2 + 4y^2 = 4$:
   (a) Draw the level sets of f to estimate the locations of maxima and minima on the constraint curve.
   (b) Use Lagrange multipliers to find the maxima and minima of f subject to the constraint.
   (c) Repeat the above two questions withe the function $f(x, y) = x^4 + y^4$ and the same constraint.
3. [Elliot] Explain in plain language why Lagrange multipliers work. That is, why do the extrema of a function f subject to the constraint g have the gradients $\vec{\nabla} g$ and $\vec{\nabla} f$ proportional to each other?
4. [Ryan] Describe the process of finding the global maximum of a function $f(x, y)$ on a square (both the interior and the boundary) to a fellow classmate who is struggling to apply the techniques.
5. [Elliot] Describe the process of finding the maxima of a function $f(x, y)$ on the hemisphere $x^2+y^2+z^2 \leq 1$ and $z \geq 0$.

2 Multiple Integrals

1. [Elliot] True or false?: The integrals
$$\int_0^1 \int_0^{\sqrt{1-x^2}} \mathrm{d}y \, \mathrm{d}x$$
and
$$\int_{0^1} \int_0^{\pi} \mathrm{d}r \, \mathrm{d}\theta$$
give the area of the same region? Justify.
2. [Elliott] True or flase?: The integral
$$\iint_D f(x, y) - g(x, y) \, \mathrm{d}x \, \mathrm{d}y$$
is the volume of the solid above a region D in the xy plane and between the surfaces $z = f(x, y)$ and $z = g(x, y)$ (where $f(x, y) > g(x, y)$). Justify.
3. [Elliot] Evaluate
$$\int_0^3 \int_0^2 x^3 y^2 \, \mathrm{d}y \, \mathrm{d}x.$$
4. [Elliot] Consider the region $y \geq x^2$ and $y \leq 2 - x^2$. Write the area of this region as an integral in both possible orders of integration. Evaluate both integrals and check that they agree.
5. [Ryan] Evaluate
$$\iint_R 2 \, \mathrm{d}A$$
where R is the region bounded between $x = |y|$ and $3x = 10 - 4y + 3y^2$.

6. [Ryan] Calculate

$$\int_0^1 \int_x^1 e^{y^2} \, \mathrm{d}y \, \mathrm{d}x$$

7. [Ryan] Let $\vec{\mathbf{r}}(t) = (f(t), g(t))$ be a differentiable plane curve whose tangent vectors are nonzero for all time, in particular, $g'(t)$ is always nonzero. Write a formula for the area between the curve $\vec{\mathbf{r}}(t)$, its tangent line at $t = 0$ and the lines $y = a$ and $y = b$.

8. [Elliot] Suppose $f(y)$ is continuous. Show that

$$\int_0^1 \int_0^1 \int_{x^2}^{x} xzf(y) \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}z = \frac{1}{4} \int_0^1 (y - y^2) f(y) \, \mathrm{d}y$$

9. [Elliot] Find the average distance of a point in a solid ball of radius a to the center of the ball.

10. [Ryan] Set up the integral for the surface area of the surface $z = f(x, y)$ over a region D in the xy plane.

11. [Ryan] Evaluate

$$\int_0^2 \int_{y/\sqrt{3}}^{\sqrt{1-y^2}} \mathrm{d}x \, \mathrm{d}y$$

12. [Textbook] Express the integral $\iiint_E f(x, y, z) \, \mathrm{d}V$ in two different orders, where E is as in fig. 1.

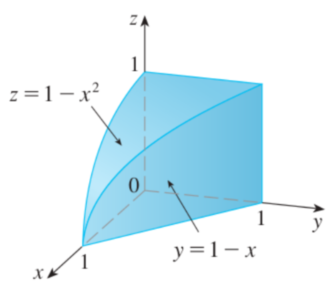

Figure 1

13. [Ryan] The curve $\vec{\mathbf{r}}(t) = (t^3 + t^2 - t - 1, t^3 - t)$ goes through the origin at $t = -1$ and $t = 1$. Set up an integral in polar coordinates to measure the area inside the loop.

14. [Ryan] Sketch the solid in spherical coordinates given by

$$1 \leq \rho \leq 2 \qquad \pi/4 \leq \phi \leq \pi/2 \qquad 0 \leq \theta \leq 2\pi$$

3 Change of Variables

1. [Elliot] Suppose you want to compute the triple integral of $f(x, y, z)$ over a cylinder rotated 45° from vertical in the yz plane. Specifically, the region of integration is $x^2 + (y + z)^2 \leq 1$ and $y - z \geq 0$ and $y - z \leq 1$.

   (a) Write down a coordinate system such that this region is a rectangle, meaning the integral of f over the region is given by a triple integral with constant bounds.

(b) Write down the triple integral of f in that coordinate system (i.e. find the correct bounds and the Jacobian).

2. [Ryan] Derive the change of coordinates formula for spherical coordinates.

3. [Ryan] Compute

$$\int_0^2 \int_{\frac{-1}{2}\sqrt{1-(x-1)^2}}^{\frac{1}{2}\sqrt{1-(x-1)^2}} \int_{\frac{x}{2}+2\frac{y^2}{x}} x \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$$