Fun/Challenge Problems

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1 Comparing Spheres

Suppose you are given twelve identical-looking balls. Eleven of them have the same weight. One ball weighs differently, but you don't know if it weighs more or less than the others. You are also given a scale. On this scale you can compare any number of balls to each other, and the output is either they weigh the same, the left side is heavier, or the right side is heavier.

What is the smallest number of steps (comparisons using the scale) it would take to figure out which ball is different?

Challenge: How many steps does it take if, in addition, you have to find out whether the different ball is heavier or lighter?

Further Connections: Information theory is an area of math that can help us understand this problem by quantifying the entropy of information, and finding out how much information we can gain at each step. For example, see this post. For an introductory reading, try Information Theory, Inference, and Learning Algorithms by David McKay.

2 The Matrix Determinant, Areas, and Lengths

A 2-by-2 matrix M

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

gives a map $\mathbb{R}^2 \to \mathbb{R}^2$, defined by

$$M\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}.$$

The **determinant** of the matrix is defined as

$$\det(M) = ad - bc.$$

1. Suppose we have the curve given by

$$x(t) = \cos(t) \qquad y(t) = \sin(t) \qquad 0 \le t \le 2\pi.$$

   (a) Compute the area contained within the curve.

   (b) Show that the area enclosed by $M\left(\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\right)$ is of the form $\det(M)\cdot A$ where A is the area from part (a). Conclude that M stretches the area enclosed by a circle by a factor of $\det M$.

   (c) Reparametrize the curve into a polar function $r(\theta)$. Now, find the area underneath the polar curve using polar area under the curve. Again, show that the area is $\det(M)\cdot A$.

2. Now, for the curve defined by

$$x(t) = \begin{cases} 0 & 0 \le t \le 1 \\ t-1 & 1 \le t \le 2 \\ 1 & 2 \le t \le 3 \\ 4-t & 3 \le t \le 4 \end{cases} \qquad y(t) = \begin{cases} t & 0 \le t \le 1 \\ 1 & 1 \le t \le 2 \\ 3-t & 2 \le t \le 3 \\ 0 & 3 \le t \le 4 \end{cases} \qquad 0 \le t \le 4.$$

   (a) Compute the area inside the loop of the curve.

   (b) Show that the area enclosed by $M\left(\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\right)$ and the x-axis is of the form $\det(M)\cdot A$ where A is the area from part (a). Conclude that M stretches the area of a square by a factor of $\det(M)$.

   (c) Reparametrize the curve into a polar function $r(\theta)$. Now, find the area underneath the polar curve using polar area under the curve. Again, show that the area is $\det(M)\cdot A$.

3. Suppose that

$$M = \begin{bmatrix} 1 & -2 \\ 3 & -6 \end{bmatrix}$$

   (a) For the curve in (1), what is the area contained by the curve $M\left(\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\right)$?

   (b) Draw the curve $M\left(\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\right)$. Why is the area contained in this curve zero?

   (c) For the curve in (2), what is the area contained by the curve $M\left(\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\right)$?

   (d) Draw the curve $M\left(\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\right)$. Why is the area contained in this curve zero?

4. *Challenge*: Show that for general $x(t)$ and $y(t)$ such that $x(0) = x(1)$ and $y(0) = y(1)$ that the area under the curve contained by
$$M\left(\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\right)$$
is
$$\det(M) \cdot A$$
where A is the area enclosed by $x(t)$ and $y(t)$ from time $0 \leq t \leq 1$.

5. Now we will investigate how M modifies arc-lengths. Suppose
$$M = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$
   (a) Compute the length of the curve in part (2).

   (b) Now, compute the length of the curve in part (2) after transforming by M and compare the two. Do they satisfy the same relation as the area?

   (c) Show that there is no function f that depends only on $\det(M)$ such that the length of the transformed curve L' is equal to $f(\det(M)) \cdot L$, where L is the length of the curve before the transformation.

   (d) Conclude that the determinant of a matrix does not give us information about curve lengths after transformations.

Further Connections: the connections between multivariable calculus and linear algebra!! It turns out that the determinant of a matrix is a great way of measuring volumes, and it motivates many formulas for things such as change of variables formulas and surface integrals. The same way we linearize problems in 1-variable calculus (for example, the derivative is the local linearization of a function), we also linearize problems in multi-variable calculus. The equivalent notion is generally a matrix. Many multi-variable problems can be understood without thinking about linear algebra as such, but linear algebra is under the hood of many interesting concepts!

3 The Grumpy Gallery Owner

An art gallery owner hired someone to hang all the paintings for an exhibit. When the person was done hanging the paintings, the art gallery owner inspected the work and found that each painting was only hanging on one nail. He was furious that each of his expensive pieces of art was relying on one single nail, so he told the person to re-hang them all on two nails each.

The worker was so annoyed that he decided to hang each painting on two nails such that if either nail was removed, the whole painting would come crashing down (just as if it were on a single nail!). How did the worker do this? [Note: this is not a physics problem]

Challenge: what if the art gallery owner demanded that each painting hang on n nails. Could the worker still hang each painting such that if any one of the n nails was removed, the whole painting would still come crashing down?

Further Connections: Topology! This problem can be solved fairly intuitively by using what is called the fundamental group of a space, in this case $\mathbb{R}^2$ with two (or n) holes in it. Techniques from topology can also be used to show other interesting facts, such as that there always are two points on earth that are antipodal and that have the exact same temperature and pressure.

4 100 Prisoners and a Light Bulb

Credit: this website/forum of riddles by William Wu.

100 prisoners are imprisoned in solitary cells. Each cell is windowless and soundproof. There's a central living room with one light bulb; the bulb is initially off. No prisoner can see the light bulb from his or her own cell. Each day, the warden picks a prisoner equally at random, and that prisoner visits the central living room; at the end of the day the prisoner is returned to his cell. While in the living room, the prisoner can toggle the bulb if he or she wishes. Also, the prisoner has the option of asserting the claim that all 100 prisoners have been to the living room. If this assertion is false (that is, some prisoners still haven't been to the living room), all 100 prisoners will be shot for their stupidity. However, if it is indeed true, all prisoners are set free and inducted into MENSA, since the world can always use more smart people. Thus, the assertion should only be made if the prisoner is 100% certain of its validity.

Before this whole procedure begins, the prisoners are allowed to get together in the courtyard to discuss a plan. What is the optimal plan they can agree on, so that eventually, someone will make a correct assertion?

Further Connections This falls in the general area of "algorithms," which is a general field of very active research. Finding the optimal solution is (as far as I know) an open problem. The expected time until all prisoners are free can either be simulated using a computer or computed using probability theory. If you have a solution and want to see how good it is, I encourage you to code it up and run it a few times to see how many "days" it takes for the prisoners to get out!