Graduate Seminar on Advanced Topics in PDE (S5B1)

Homogenization of Partial Differential Equations

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Although seemingly homogeneous to the naked eye, most materials are heterogeneous on small scales. When modelling such a physical situation, one commonly derives a PDE resolving the microscopic scale, leading to a dependence on a small parameter $\varepsilon > 0$. This ε-dependent PDE is usually difficult to study as its solution oscillates on the small scale ε. The aim of homogenization is to understand the "effective" large scale behaviour in the limit $\varepsilon \to 0$.

A concrete example is a material made up of two components with electrical conductances A_1 and A_2, respectively. Suppose this "mixture" happens on the microscopic scale ε, for example by assuming the conductance matrix to be of the form $a_\varepsilon(x) = \chi(\frac{x}{\varepsilon})A_1+(1-\chi(\frac{x}{\varepsilon}))A_2$ for some 1-periodic characteristic function χ. Then our question of homogenization amounts to studying the behaviour as $\varepsilon \to 0$ of the solutions u_ε to the uniformly elliptic PDEs

$$-\nabla \cdot \big(a_\varepsilon \nabla u_\varepsilon\big) = f$$

with source f (and some boundary conditions). In this particular case, one can show that homogenization takes place, i..e, the limit $u_0 := \lim_{\varepsilon\to 0} u_\varepsilon$ exists and u_0 solves a much simpler constant-coefficient PDE of the form

$$-\nabla \cdot \big(a_0 \nabla u_0\big) = f,$$

where the effective coefficient field a_0 has to be determined. In the one-dimensional case, a_0 is simply the harmonic (!) mean of $a_\varepsilon|_{\varepsilon=1}$. In higher dimensions, it is given implicitly by yet another PDE—the "cell problem"—, which furthermore gives insight into the oscillations of u_ε on the microscopic scale ε.

In this compact 3-day seminar, we will discuss the homogenization of several types of PDEs, ranging from the elliptic case explained above to Hamilton-Jacobi and hyperbolic equations. Along the way, we will learn a wide range of analytical techniques to prove (or disprove) that homogenization takes place.

Prerequisites: Working knowledge in Functional Analysis and PDEs.

Venue: March 25–27. Exact times will be decided during the preliminary meeting.

Preliminary meeting: 13–14 on Tuesday, January 28 in N0.008.