Loop Groups and G-bundles on curves

Constantin Teleman*

Introduction

This lecture surveys some results and methods in the study of the moduli of algebraic principal bundles over a complex curve Σ, structured by a complex reductive group G.[1] Although I touch upon questions dealing purely with coherent sheaf cohomology, I will focus on a topological aspect, namely the *fundamental classes* of the stack $\mathfrak{M}$ and the moduli space M of principal G-bundles—that is, the problem of *integration* over those objects. A compelling answer emerges in *topological K-theory*, rather than in cohomology. It was discovered using twisted K-theory, but its proof, which is joint work with C. Woodward [TW], uses more traditional methods of sheaf cohomology. As a concrete application of the new results, I will sketch our proof of the Newstead-Ramanan conjecture for arbitrary G.

I will only discuss smooth curves, but more recent developments (Gieseker, Nagaraj-Seshadri, Kausz) allow one to extend some of the results to stable curves, at least when $G = GL_N$. (The outline of the theory for general groups is clear, but a more refined understanding of compactifications of G seems needed.) Varying the curve leads to the *Gromov-Witten theory of BG*, and one can reasonably expect that equivariant Gromov-Witten theory of projective varieties can be understood by this route. Whether the new insights can be applied to G-bundles over a complex surface is unceertain, but worth pursuing.

Three moral lessons profess themselves, even when studying as simple a moduli problem as M. While they are well-known to experts, they are sometimes ignored in the literature, and one suspects that some of the complexity encountered in certain questions of topological intersection theory on various moduli spaces is caused by that fact alone. The lessons are

(i) K-theory is better than cohomology;

(ii) Stacks are better than spaces;

(iii) Symmetry.

Lesson (i) applies anytime groups are acting, and the logic is (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i): the symmetry of the question imposes strong constraints on the answer, passing from the stack to the space discards information and breaks some of the symmetry, and K-theory is forced upon us because the cohomological fundamental homology class of a stack, if defined at all, tends to break some symmetries that the K-homology class remembers. In fact, I do not know how to define a homology fundamental class of a moduli stack that differs from that of the space, since in typical GIT situations, the difference between the two has positive codimension. One can illustrate the problem with the stack BG itself: when G is not finite, no sensible "integration" over BG presents itself, whereas the functor $V \mapsto V^G$ of invariant vectors, from representations of G to

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[1]For explicit formulae such as abelian reduction, it helps to assume that G is connected and $\pi_1 G$ is free.

vector spaces, gives a good candidate for a "direct image map" $K_G(\text{point}) = \text{Rep}(G) \to \mathbb{Z}$ which satisfies a semblance of Poincaré duality via the non-degenerate pairing

$$\text{Rep}(G) \times \text{Rep}(G) \to \mathbb{Z}, \qquad (V, W) \mapsto \dim(V \otimes W)^G.$$

When dealing with *spaces*, there is no substantive difference between knowledge of rational cohomology and that of rational K-theory, because the Chern character $ch : K^*(X; \mathbb{Q}) \to H^*(X; \mathbb{Q})$ is a ring isomorphism. If anything, K-theory is preferable in the following sense. When X is a compact almost complex manifold, K-theory integration (the "topological index") is expressed in terms of the cohomological one by the topological Riemann-Roch formula,

$$\text{Ind}(X; V) = \int_X \text{Td}(X) \wedge ch(V).$$

This can be difficult to use in examples, as the experience with moduli of G-bundles shows. In the reverse direction, integration is more easily recoverable as a leading term in an index

$$n^{-\dim X} \cdot \text{Ind}(X; \psi^n V) = \int_X ch(V) + O(1/n),$$

where ψ^n is the nth Adams operation (the additive operation satisfying $\psi^n L = L^n$ on line bundles); see [TW, §5] for a recent illustration of this reverse calculation.

1. Loop groups and G-bundles

The relation between G-bundles on Σ and the *smooth loop group* LG of C^∞ maps from the circle to G stems from "Segal's double coseet construction" of [PS] (attributed there to Atiyah). Let $\Delta \subset \Sigma$ be a disk bounded by a smooth circle and let $\Sigma^\circ := \Sigma \setminus \Delta$; denote by $G[\Delta]$ and $G[\Sigma^\circ]$ the respective groups of holomorphic maps to G with smooth boundary values. We then have

$$\mathfrak{M} := G[\Delta] \backslash LG / G[\Sigma^\circ] = \left\{ \begin{array}{c} \text{isomorphism classses of} \\ \text{holomorphic } G\text{-bundles} \end{array} \right\}. \tag{1.1}$$

The loop in the group serves as a transition function between the trivial bundles on the two sides of the circle.[2] Letting X_Δ and X_{Σ° be the single cosets of LG by the respective groups $G[\Delta]$ and $G[\Sigma^\circ]$, the double coset space is also the quotient $(X_\Delta \times X_{\Sigma^\circ})/LG$.

The equality (1.1) is *a priori* one of sets, but in fact it represents an isomorphism of *analytic stacks*: this means that the functor which assigns to an analytic space S the category of G-bundles over $S \times \Sigma$ is equivalent to that of morphisms into the quotient stack $(X_\Delta \times X_{\Sigma^\circ})/LG$. The latter functor assigns to a space S the category of pairs consisting of a principal LG-bundle over S and an LG-map to $X_\Delta \times X_{\Sigma^\circ}$, and there is a similar description for morphisms into the double coset stack (1.1).

The varieties X_Δ and X_{Σ° carry natural ample line bundles (determinants of cohomology along Σ in representations of G), which are projectively equivariant for LG. Appropriate products of such line bundles are genuinely equivariant (the projective cocycles cancel), and one can define a notion of "stability" for the action, by a Hilbert-Mumford criterion, leading to a geometric invariant theory quotient which is none other than the moduli space of semi-stable G-bundle of Narasimhan, Seshadri and Ramanathan. In fact, the entire structure one usually associates to linearised reductive group actions on projective manifolds can be applied to the action of LG on the product $X_\Delta \times X_{\Sigma^\circ}$, the stratification of Ness, Kirwan and Hesselink, has an analogue in this situation, and is none other than the Shatz, or Narasimhan-Seshadri. The

[2] With real-analytic loops, the assertion follows immediately from the triviality of holomorphic bundles on the two halves of Σ, but patching the smooth structures along smooth loop requires a small extra argument.

usual way to present this information uses the Atiyah-Bott picture [AB] of bundles as $\mathfrak{g}$-valued $(0,1)$-connections on Σ modulo complex gauge transformations, but the result can be described entirely in terms of the stack $\mathfrak{M}$, and from there it lifts to $X_\Delta \times X_{\Sigma^\circ}$ by the equivalence of stacks just described [T2, §9].

The topological implications of this structure were exploited with spectacular success by Atiyah and Bott, and generalised to arbitrary GIT settings $X//G$ by Frances Kirwan [K]. In particular, the stratification is *equivariantly perfect* over the rationals, which mean (in new language) that the filtration induced on the rational cohomology $H^*(\mathfrak{M};\mathbb{Q})$ has, as its associated graded space, the direct sum of the cohomologies of the strata (shifted by a Thom isomorphism).[3] In the special case when all semi-stable points have finite stabilisers, the rational cohomology $H^*(X//G;\mathbb{Q})$ of the GIT quotient orbifold is isomorphic to the equivariant cohomology $H^*_G(X^{ss};\mathbb{Q})$ of the semi-stable stratum, leading to Kirwan's famous surjection $H^*_G(X;\mathbb{Q}) \to H^*(X//G;\mathbb{Q})$. This was used to determine the Betti numbers of GIT quotients, in particular, in [AB], for the moduli space of stable vector bundles over Σ.

A clean description of the kernel of Kirwan's map, and hence of the algebra structure of $H^*(X//G;\mathbb{Q})$, in terms of data about X and the group action took longer to achieve. I will return to this question below in the context of K-theory.

2. Coherent sheaf cohomology

K-theory is closely related to the index of elliptic differential operators: the *direct image* map in K-theory (the integral of a vector bundle along a compact Spin manifold) is the index of the Dirac operator acting on sections of the bundle. For holomorphic vector bundles over complex manifolds, the index is described more elegantly as the alternating sum of the (dimensions of) coherent sheaf cohomology groups.[4] So there is a close connection between topological K-theory and coherent sheaf cohomology, which we will exploit in the case of $\mathfrak{M}$.

(2.1) Algebraic flag varieties. To do so, a slight variation from §1 is needed; this is because we are not yet in a position to study coherent sheaf cohomology over infinite-dimensional manifolds such as X_Δ.[5] Thankfully, there are algebraic versions of X_Δ and X_{Σ°. They are defined from the formal Laurent loop group $G(\!(z)\!)$, replacing LG, its subgroup of formal Taylor loops $G[[z]]$, replacing $G[\Delta]$, and the group $G[\Sigma^\times]$ of algebraic G-valued functions on $\Sigma^\times$. Here, z is a formal coordinate at the centre C of the disk and $\Sigma^\times = \Sigma \setminus \{C\}$. Then, $X_0 := G(\!(z)\!)/G[[z]]$ is an ind-projective variety, sometimes called the *loop Grassmannian* of G; it admits a global embedding in an direct limit projective space [Ku]. On the other hand, $G(\!(z)\!)/G[\Sigma^\times]$ is a scheme of infinite type (but reasonably 'tame, as it is covered by products of smooth finite-dimensional varieties and infinite-dimensional affine spaces). The double coset description or $\mathfrak{M}$ continues to apply, in the world of stacks over the category of algebraic varieties.[6]

The loop Grassmannian X_0 was extensively studied; early results [Ma, Ku] established the anticipated purity of cohomology for irreducible homogeneous vector bundles, as one could expect from the Borel-Weil-Bott theorem for reductive groups. In this respect, X_0 behaves like a generalised flag variety for the loop group $G(\!(z)\!)$. Later work on D-modules by Kashiwara-Tanisaki and by Beilinson and collaborators continued to find similarities between the loop

[3] That is, the equivariant Gysin sequence collapses.

[4] A twist by $\sqrt{K}$ is involved when translating the Dirac picture into the holomorphic one; this shows up as a "quantum" shift $h \mapsto h + c$ by the dual Coxeter number c in the first Chern class of line bundles over $\mathfrak{M}$, in character and the Verlinde-related formulas.

[5] Recent progress by Lempert and his students has brought this closer to our reach, but even so only simple examples seem presently accessible.

[6] This works well only when G is semi-simple; the general reductive case requires some changes.

Grassmannian and its finite-dimensional counterparts.[7]

(2.2) The role of thick flag varieties. By contrast, the "thick" flag varieties $X_{\Sigma^\times}$ received less attention, one of the first cohomological results being the Borel-Weil-Bott theorems of [T1]. One reason is that, unlike the ind-variety X_0, the thick variety is not directly amenable to finite-dimensional techniques, and new methods were needed to establish cohomology vanishing results that form the basis of most calculations. A short survey of the initial results is [T3]. The model here is the Borel-Weil-Bott theorem for generalised flag varieties G/P, which describes the cohomology of vector bundles associated to irreducible representations of P. (G replaces the loop group and P replaces $G[\Sigma^\times]$.) Thus, the cohomology of line bundles over $X_{\Sigma^\times}$ lives in a single degree, and the same holds for homogeneous vector bundles associated to irreducible representations of $G[\Sigma^\times]$.

As one would expect, the non-zero cohomology group is a highest-weight representation of $G((z))$; unlike the case of X_0, this is now *reducible*, but the co-factor of each irreducible representation can be identified with a *space of conformal blocks* in the WZW model. Proving the "Verlinde dimension" formula for these co-factors [V] motivated a great deal of the initial research into $\mathfrak{M}$ and the vector bundles over it. The original proof [TUY] of the "factorisation formula" for the dimension had to be supplemented by the determination of the "3-point function" [**?**], but also by foundational results [BL, DS, LS] pertaining to the algebraic version of the double coset formula in §1.

While the higher cohomologies over $\mathfrak{M}$ were described in [T1], progress toward understanding them and the relation between the stack $\mathfrak{M}$ and the moduli space M

(2.3) Coherent sheaf cohomology analogues of Kirwan's results. It was a surprise (to the author of the present note, among others) that Kirwan's topological understanding of the GIT stratification of linearised G-varieties had immediate implication for the cohomology of algebraic line and vector bundles. It seems that only the right example was missing, and this was provided by $\mathfrak{M}$.

(2.4) Differential forms, D-modules, Higgs bundles.

3. Index theory over $\mathfrak{M}$

Recall that Kirwan showed the surjectivity of the restriction map in rational equivariant cohomology, $H^*_G(X;\mathbb{Q}) \to H^*_G(X^{ss};\mathbb{Q})$. In the case of orbifold GIT quotients $X//G$, the problem of specifying the ring structure on the rational cohomology of $X//G$ in terms of the equivariant cohomology of X reduces, by Poincaré duality, to that of describing the integration over $X//G$ of equivariant cohomology classes on X. This was answered in several stages and variations [JK, M, Me]. In particular, in the case of G-bundles, Witten [W] wrote down an explicit answer derived by (non-rigorous) path-integral arguments. While this was later justified by cohomological methods in special cases [JK] and later (but independently of [TW]) in general [**?**], the answer in K-theory is structurally sounder and seems to provides an explanation for Witten's formula, as opposed to a verification: Witten's sum ranges over the support of the Chern character in twisted equivariant K-theory, in the limit when the twisting becomes infinite. This is the same large ψ^n limit which converts the index to an integral in the introduction.

(3.1) Finiteness results. The following loose statement captures the key properties of $\mathfrak{M}$ and M, insofar as K-theory integration goes. For precise statements I refer to [T2] and [TW].

[7] However, a striking dissonance was found in [FGT]. It appears that the viewpoint which takes X_0 to be compact has its limitations, related to famous combinatorial identities.

3.2 Theorem. *"Index theory works on the stack $\mathfrak{M}$": the coherent sheaf cohomologies of Atiyah-Bott bundles are finite-dimensional and vanish in high degree. Hecke correspondences on $\mathfrak{M}$ (defined by its decorated versions, bundles with parabolic structures) interact in predictable ways with the index. For any Atiyah-Bott class, the indexes over the stack and the space agree, after a large line bundle twist.*

Effective bounds for *high* and *large* can be specified. The "predictions" come from the Leray spectral sequences; finiteness makes those calculations meaningful.

(3.3) The index formula. The main result of [TW] reduces the index problem over $\mathfrak{M}$ to the one over the stack $\mathfrak{M}_T$ of T-bundles, where $T \subset G$ is the maximal torus. Strictly speaking, this applies only to connected G with free π_1, as in general there are additional contributions from principal bundles under subgroups of the normaliser of T in G. The assertion applies to a certain "tautological" subring of $K^*(\mathfrak{M})$. The subring is large enough to surject onto every component of $K^*(\mathfrak{M}^{ss})$, and is in fact dense in $K^*(\mathfrak{M})$ for the inverse limit topology defined by the Atiyah-Bott strata.[8] The question of describing "the largest natural ring" of K-classes to which the theorem applies does not have a satisfactory answer at present; it is very relevant to the Gromov-Witten theory of quotient stacks.

To state the theorem, let E^*V be the vector bundle over $\mathfrak{M} \times \Sigma$ associated to the universal principal G-bundle and a G-representation V. For $x \in \Sigma$, call E^*_xV the restriction of E^*V to $\{x\} \times \mathfrak{M}$; its topological class is independent of x. Moreover, let $E^*_\Sigma V$ be the total direct image of $E^*V \otimes K_\Sigma^{1/2}$ along Σ. Finally, let $\mathcal{L}$ be any line bundle which is a root of an inverse determinant[9] of some $E^*_\Sigma V$. These classes are the *Atiyah-Bott generators.* A *tautological K-class* on $\mathfrak{M}$ is a polynomial in the Atiyah-Bott generators.

3.4 Theorem (Abelian reduction)**.** *The index over $\mathfrak{M}$ of a tautological class $\mathcal{E}$ equals the index of $\mathcal{E}/\lambda_{-1}\nu^*$ over $\mathfrak{M}_T$, divided by the order of the Weyl group.*

All components of $\mathfrak{M}_T$ admit T as a group of automorphisms. Consequently, "integration over $\mathfrak{M}_T$" involves extraction of the T-invariant vectors, which on characters is realised by integration over (the compact part of) T. However, the character of $\mathcal{E}/\lambda_{-1}\nu^*$ is a rational function; the correct interpretation of its integral is given in [TW]: summation over the components of $\mathfrak{M}_T$ must be done before integration. The answer can be summed in closed form; a sample formula appears in the next section.

(3.5) Families of curves.

(3.6) Gromov-Witten theory of X/G?

4. Application: the Newstead-Ramanan conjecture

The conjecture, originally stated for rank 2 bundles of odd determinant, predicts the vanishing of the top $(g-1)\cdot\ell$ rational Chern classes of TM, where ℓ is the rank of G and M a smooth, proper moduli space of principal G-bundles over Σ.

To get a smooth (or orbifold) moduli space, we must either restrict ourselves to $G = GL_N$ or $\mathbb{P}GL_N$, and the components of moduli space parametrising bundles of degree prime to N, or else add a decoration, in the form of a Borel structure at some point of the curve, and use a generic polarisation to define the moduli space. The conjecture was proved for rank 2 bundles of odd degree by Gieseker [G] and Zagier [Z], and in rank 3 by Kiem-Li [KL].[10]

[8] Unfortunately, I know of no written account of this at present.

[9] We invert because $\det E^*_\Sigma V$ is negative.

[10] Recent progress by Brion and Kausz holds the promise for an argument for arbitrary rank.

Here, I outline a general argument based on the abelian reduction of the index over $\mathfrak{M}$. However, I will spell out the calculation only for $G = SL_2$, but this captures all that is needed for the general case. The full argument will appear in a joint paper with C. Woodward.

We would like to argue as follows. Letting $\pi : \Sigma \times \mathfrak{M} \to \mathfrak{M}$ be the projection and ad_G the universal Lie algebra bundle over the product, TM is the restriciton to M of the complex $R\pi_*(\mathrm{ad}_G)$. However, Poincaré duality (which applies whenever M is an orbifold) and the abelian reduction formula show that K^0-classes over M which come from $\mathfrak{M}$, such as $[TM]$, are detected by their restriction to the moduli stack $\mathfrak{M}_T$ of T-reduced bundles. Now, over T-reduced bundles, ad_G has the trivial summand $\mathfrak{t}$, whose index is a trivial bundle of rank $(g-1)\cdot\ell$. This summand should account for the vanishing of the top Chern classes, in that number.

As it stands, this argument is faulty, because $R\pi_*(\mathrm{ad}_G)$ is a complex, a *virtual bundle* in K-theory, and its virtual dimension has no bearing on vanishing of Chern classes; in fact, over $\mathfrak{M}$, the Chern classes of the tangent complex do not vanish. Nonetheless, it is not hard to evade the trap. In general, the vanishing of the top d rational Chern classes of a vector bundle V of rank r over a space X is equivalent to the "existence of a trivial summand of rank d for V" in rational K-theory. More precisely, it means that the topological classifying map $X \to BU(r)$ for V deforms rationally to a map to $BU(r-d) \subset BU(r)$. Letting $\lambda^k(V)$ be the K^0-class of $\Lambda^k V$ and considering

$$\lambda_t(V) := 1 + t\lambda^1(V) + t^2\lambda^2(V) + \cdots \in K^0(X;\mathbb{Q})[t],$$

the total Lambda-class of V in the ring of polynomials with coefficients in rational K-theory, the vanishing is equivalent to the statement that $(1+t)^d$ divides $\lambda_t(V)$ in that same ring. When X satisfies rational Poincaré duality, it suffices to check that the index pairing of $\lambda_t(V)$ with any other K-class vanishes to order d at $t = -1$. In the case at hand, with $X = M$ and $V = TM$, we can write an explicit formula for these pairings using the summation formula from [TW]. We take for simplicity $G = SL_2$ and restrict attention to the index of the bundles

$$\lambda_t(T\mathfrak{M}) \otimes \mathcal{O}(h) \otimes E_x^* U \tag{4.1}$$

where $\mathcal{O}(h)$ is the line bundle with Chern class $h \geq 0$. Other K^0-generators do not alter the substance of the argument (Remark 4.4).

A result of [T2, §7] ensures that the indexes over $\mathfrak{M}$ and M agree, for sufficiently large h (depending on U). We obtain from [TW] the expression

$$\sum_u (1+t)^{g-1} \cdot \left[\frac{(1+tu^2)(1+tu^{-2})}{(u-u^{-1})^2}\right]^{g-1} \cdot \left[2h + 4 - 4t\left(\frac{u^2}{1+tu^2} + \frac{u^{-2}}{1+tu^{-2}}\right)\right]^{g-1}$$

with u ranging over the solutions of the equation depending on the parameter t

$$u^{2h+4}\left[\frac{1+tu^{-2}}{1+tu^2}\right]^2 = 1. \tag{4.2}$$

The solutions must be 'tracked' from the roots of unity with strictly positive imaginary part, which one finds when $t = 0$.

The factor $(1+t)^{g-1}$, which carries the desired vanishing, is the "trivial summand" in $T\mathfrak{M}$ over $\mathfrak{M}_T$. We must now verify that the rest of the formula does not conspire to create a pole at $t = -1$. This requires a bit of care; in fact, as $t \to -1$, two pairs of solutions u_t approach the singular points ± 1, from opposite sides of the real axis, which is cause for concern. However, the numerator at $t = -1$ in middle factor turns out to cancel the singularity in the Weyl denominator. Verifying this requires us to relate the rates of convergence of t and u. Roughly, since two roots u approach each singularity as $t \to -1$, they converge as the square root of $t + 1$. Assuming that, a leading-order calculation shows the regularity of both questionable factors in the index formula. The reader is invited to check this last fact; here, I verify the rate-of-convergence.

We break up (4.2) into the two equations

$$u^{h+2}(1+tu^{-2}) = \varepsilon(1+tu^2), \qquad \varepsilon = \pm 1. \tag{4.3}$$

Set now $t=-1$. When h is even and $\varepsilon=-1$, we have double roots at $u=\pm1$. These are "new" roots: for $\varepsilon=+1$, the two simple "old" roots of unity at $u=\pm1$ are valid for any t in (4.2). For odd h, we get a "new" double root at $u=-\varepsilon$ and the old simple root of unity at $u=\varepsilon$. Express now

$$t = \frac{\varepsilon - u^{h+2}}{u^h - \varepsilon u^2}$$

and set $u=\pm(1+x)$ if h is even and $u=-\varepsilon(1+x)$ for odd h. For h even and $\varepsilon=-1$ we get

$$t = \frac{-1-1-(h+2)x - O(x^2)}{1+hx+O(x^2)+1+2x+O(x^2)} = -1+O(x^2),$$

as asserted, and similarly for odd h.

4.4 Remark. To complete the argument, we must include K^*-generators of the form $E^*_\Sigma V$. Insert a factor of $\exp[s\cdot E^*_\Sigma V]$ in the bundle (4.1), with a formal variable s. According to [TW], calling $\varphi(u)$ the character of V, the effect is to replace our third factor in the index formula by

$$\left[2h+4-4t\left(\frac{u^2}{1+tu^2}+\frac{u^{-2}}{1+tu^{-2}}\right)+s\ddot{\varphi}(u)\right]^{(g-1)} \tag{4.5}$$

where the dot denotes ud/du; moreover, equation (4.2) gets perturbed to

$$u^{2h+4}\left[\frac{1+tu^{-2}}{1+tu^2}\right]^2 = \exp[-s\cdot\dot{\varphi}(u)].$$

By symmetry, $\dot{\varphi}(\pm1)=0$, so the right-hand side is $1+s\cdot O(x)$. This allows the argument for the quadratic dependence of $t+1$ in $u\pm1$ to proceed. Finally, odd K^*-generators only lower the exponent $(g-1)$ in (4.5), and do not cause a singularity.

4.6 Remark. For a general group, the first factor will be $(1+t)^{(g-1)\ell}$, the middle factor is $\prod_\alpha(1+te^\alpha)(1-e^\alpha)^{-1}$, ranging over all roots α, while the third factor contains the sum

$$\frac{e^\alpha}{1+te^\alpha}+\frac{e^{-\alpha}}{1+te^{-\alpha}}$$

over all positive roots. The values of this expression are to be summed over the solutions of a system of equations analogous to (4.2) (one equation for each simple co-root), and the argument for regularity as $t\to-1$ is nearly unchanged.

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