

Topological Field Theories in 2 dimensions

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Abstract

We discuss a formality result for 2-dimensional topological field theories which are based on a semi-simple Frobenius algebra: namely, when sufficiently constrained by structural axioms, the complete theory is determined by the Frobenius algebra and the grading information. The structural constraints apply to Gromov-Witten theory of a variety whose quantum cohomology is semi-simple. Some open questions about semi-simple field theories are mentioned in the final section.

1. Introduction

The notion of a *Topological Field theory* (TFT) was formalised by Atiyah and Witten [W] and modelled on Graeme Segal's definition of a *Conformal Field theory*. It was intended to supply the structural framework for the new topological invariants of the 1980's, in particular Donaldson theory and 3-dimensional Chern-Simons theory.

The distinguishing feature of the new invariants was their *multiplicativity* under unions, rather than the additivity common to classical algebraic topology invariants, such as characteristic classes. The source of additivity is the *Mayer-Vietoris sequence* for homology.

Quantum field theory explains this behaviour heuristically: the invariants of a manifold X are integrals, not over X , but over a space of *fields* on X , these fields being maps to another, fixed space. The space of fields is “multiplicative in pieces of X .” Formulated naïvely, this can only be carried out for target spaces X with strong finiteness properties (such as finite sets, or finite groupoids, leading to gauge theory for finite groups), but good use of geometric or analytic information can lead to a notion of integration over more interesting spaces of fields. This happens for instance in Gromov-Witten theory, where a fundamental class for integration in the space of continuous maps from a closed surface to a Kaehler manifold M is defined by the finite-dimensional cycle of holomorphic maps.

While in recent years, the notion of TFT has experienced substantial refinements, passing through the *open-closed* TFT's of Moore and Segal [MS, C], to an all-dimensional notion relying on higher category theory [Lu], the original notion is the best starting point:

1.1 Definition. A TFT is a symmetric monoidal functor from the oriented n -dimensional bordism category to the category of complex vector spaces. The monoidal structures are given by disjoint union, respectively tensor product.

Thus, to each closed oriented $(n - 1)$ -manifold (object in the n -bordism category) there is assigned a vector space, to disjoint unions we assign tensor products, to an n -dimensional bordism we assign linear maps between the boundary spaces, and the gluing of bordisms leads to the composition of linear maps.

It can be said that the interest and activity surrounding the notion of TFT has not been met by commensurate applications along the lines originally intended. In dimension 3, a classification of TFT's with special properties was given by Reshetikhin-Turaev in terms of *modular tensor categories*; the most famous example is Chern-Simons theory. By contrast, in higher dimension, there seem to be no interesting theories: all examples are built from characteristic classes. The

sophisticated 4-dimensional TFT's constructed from gauge theories and their Floer homologies, or from Khovanov's categorified knot invariants, do *not* satisfy Definition 1.1, but only a variant of it.

One striking exception is the case of surfaces ($n = 2$), where a natural extension of the notion of TFT, that of a *Cohomological Field theory*, has framed the solution of a classical enumerative question of algebraic geometry, the problem of counting curves of fixed degree and genus in a projective manifold, with specified incidence properties.

(1.2) *Two dimensions.* The classification of (compact, connected, oriented) topological surfaces has long been known: the invariants are the number of boundary components and the Euler characteristic. TFT's in dimension 2 were initially studied as a toy model and their structure was understood early on. Recall that an (associative) algebra A is called *Frobenius* if it comes equipped with a *trace* $\theta : A \rightarrow \mathbb{C}$ for which $a, b \mapsto \theta(a \cdot b)$ gives a perfect, symmetric pairing. In particular, $\dim A < \infty$.

1.3 Theorem (folklore, see [A]). *A 2-dimensional oriented TFT with vector space A assigned to the circle is equivalent to the datum of a commutative Frobenius algebra A over \mathbb{C} .*

The multiplication on A is defined by the pair of pants with two boundary circles incoming and one outgoing, and the trace by the disk with incoming boundary. Divertingly enough, in spite of this simple classification, it is in 2D that the original notion of TFT has seen powerful applications.

2. Gromov-Witten theory

My application is to *Gromov-Witten theory*, which generalises a classical and delicate question in enumerative geometry: counting algebraic curves in a projective manifold, with prescribed degree and intersection conditions.

For example, there is a unique linear map $\mathbb{P}^1 \rightarrow \mathbb{P}^n$ sending the points $0, 1, \infty$ to three linear subspaces placed in general position and with total dimension $n - 1$. GW theory encodes this information by *deforming* the ordinary cohomology algebra

$$H^*(\mathbb{P}^n) = \mathbb{C}[\omega]/\langle \omega^{n+1} \rangle.$$

into the (small) *quantum cohomology algebra*, parametrised by $q \in \mathbb{C}^*$,

$$QH^*(\mathbb{P}^n) = \mathbb{C}[\omega]/\langle \omega^{n+1} - q \rangle$$

The coefficient 1 in front of q indicates the uniqueness of the map, while its exponent 1 is the degree. Seen most naturally, q lives in $\mathbb{C}^* = H^2(\mathbb{P}^n; \mathbb{C}^*)$, which can be seen as the exponentiated image of $H^2(\mathbb{P}^n; \mathbb{C})$. The reader is probably familiar with the definition of small quantum cohomology for any projective manifold X : using the Poincaré duality bilinear form on $H := H^*(X; \mathbb{C})$, the datum of a multiplication $H \otimes H \rightarrow H$ is equivalent to that of a vector in $H^{\otimes 3}$. On three homology classes α, β, γ , this vector takes the value

$$\sum_{\delta \in H_2(X)} N_{\alpha\beta\gamma}(\delta) \cdot q^\delta, \tag{2.1}$$

where $N_{\alpha\beta\gamma}(\delta)$ is the number of maps of degree δ from \mathbb{P}^1 to X sending $0, 1, \infty$ to three cycles representing α, β, γ placed in general position. When this number is not finite, it is set to zero. The variable q lives in the torus $H^2(X; \mathbb{C}^*)$ and can be raised to powers $\delta \in H_2(X; \mathbb{Z})$.

The precise definition of the numbers $N_{\alpha\beta\gamma}(\delta)$ for general manifolds requires more sophisticated tools — construction of the moduli space of stable maps and its virtual fundamental class — which we shall overlook here. The proof that the multiplication is associative, while formally a consequence of the existence of a “four-point function”, also relies on those constructions. Convergence of the series in (2.1) seems to be an open question for general target manifolds.

(2.2) *Frobenius structure on quantum cohomology.* Just like ordinary cohomology, the quantum cohomology of a projective manifold X is a Frobenius algebra, with the (classical) Poincaré duality pairing on H . In fact, we get a family of 2D TFT’s parametrised by $q \in H^2(X; \mathbb{C}^*)$. There is a general method to extend the space of parameters to the rest of $H^{\text{ev}}(X; \mathbb{C})$, see for instance [T, §6]. This extended, “big” quantum cohomology incorporates the count of \mathbb{P}^1 ’s with arbitrarily many marked points and incidence conditions. The properties of the resulting structure on the space H were abstracted into the notion of a *Frobenius manifold* [D1, D2]; see also [G1, M1].

One important ingredient incorporates the *grading* on cohomology. Literally, this is broken by quantum multiplication — see the case of $QH^*(\mathbb{P}^n)$ above — but the grading is restored in the entire *family* of multiplications by grading the functions on the parameter space $H^{\text{ev}}(X; \mathbb{C})$: thus, one declares $\deg q^\delta = \langle c_1(X) | \delta \rangle$, using the first Chern class of X , and grades the rest of cohomological parameters by the shifted normalised degree $\deg / 2 - 1$. The quantum multiplication, when viewed as a tensor field on $H^{\text{ev}}(X; \mathbb{C})$, is homogeneous of degree 1 in this grading. Thus, $\deg q = n + 1$ in the \mathbb{P}^n example, rendering the identity defining QH^* homogeneous of degree n , once we remember that q stands for $q \cdot 1 \in QH^0$, where 1 has degree (-1) . (To explain this slightly counter-intuitive shift, see Remark 2.3 below.) Odd cohomology can be incorporated using the language of supermanifolds.

The more general Gromov-Witten invariants \overline{Z}_g^n mentioned in §3 below are also homogeneous, of prescribed weights. When spelt out, the homogeneity property encodes the fact that the virtual fundamental cycles of the moduli space of stable maps have topologically determined dimensions, depending on the degrees of the maps and on $c_1(X)$.

2.3 *Remark.* Geometrically, the grading is implemented by the *Euler vector field* on the Frobenius manifold H . Specifically, given $u \in H^{\text{ev}}(X)$, let $u' \in H^*(X)$ be obtained from u by re-scaling its degree d part by $d/2$; the value of the *Euler vector* at u is $v := c_1(X) + u'/2 - u$. Dubrovin’s general theory of Frobenius manifolds shows that, near any point with semi-simple quantum multiplication, the eigenvalues of quantum v -multiplication give a local coordinate system on the parameter space, [D2, Thm. 3.1].

The Frobenius manifold of quantum cohomology contains answers to most enumerative questions about rational curves in X , concerning incidence and tangency conditions with generally positioned cycles in X at any specified collection of marked points. (Actually, without some positivity property of the target manifold X , the curves must be counted in a virtual sense, taking account of obstructions.) A small exception concerns the case of one and two marked points, where extra information in the form of a *calibration* of the Frobenius manifold is needed [G1] (see also Remark 3.3 below). The precise formulation and explanation for this fact lies in the theorem that *genus zero cohomological field theories in two dimensions are equivalent to germs of Frobenius manifolds*. Manin’s book [M1] is largely devoted to the proof of this.

(2.4) *Givental’s Reconstruction Conjecture.* Quantum cohomology generalises to incorporate curves of any genus, with any number of marked points and incidence invariants, defined by the same method used for rational curves. The fundamental result (Ruan, Tian, Li, McDuff, Salomon) ensures that the numerically defined GW invariants are governed by the structure of an *all-genus Cohomological Field theory* (CohFT). We will briefly review that notion below, but for now let us focus on one important consequence of this structure. Recall that a commutative algebra over \mathbb{C} is called *semi-simple* if it is isomorphic to a direct sum of copies of \mathbb{C} . Semi-simple Frobenius algebras are easy to classify: the trace must be diagonal in the basis of projectors, so all we need to know are the projector traces, a collection of non-zero complex numbers. Classical cohomology of a manifold is never semi-simple, but the *quantum* multiplication of some important projective manifolds, such as projective space above, is semi-simple for generic values of the parameter (see §2.8 below for more examples).

It is for this class of manifolds that Givental formulated (and proved, in the toric Fano case) the following remarkable reconstruction conjecture:

2.5 Conjecture (Givental, [G2]). *For a compact symplectic manifold X whose quantum cohomology algebra is semi-simple at generic values of the parameter $u \in H^{ev}(X)$, all Gromov-Witten invariants are explicitly determined from genus zero information.*

Loosely speaking: counting rational curves determines the answer to enumerative questions for curves of all genera. Givental predicted a formula for the generating function of all GW invariants in his framework of quantised quadratic Hamiltonians.

2.6 Theorem ([T]). *Givental’s conjecture holds. More precisely, the GW (ancestor) invariants are determined by a recursive relation from the quantum multiplication law at a single semi-simple value of the parameter u , and from the first Chern class $c_1(X)$.*

We will review the recursion in §3 below. I refer to Givental [G2] for the discussion of ‘ancestor’ versus ‘descendent’ Gromov-Witten invariants; suffices to say here that the ancestor invariants are missing precisely the calibration of the Frobenius manifold we mentioned earlier.

2.7 Remark. Givental [G1] spells out the reconstruction using the Frobenius manifold. But his discussion shows that the knowledge of a *single* semi-simple quantum multiplication law suffices.

Theorem (2.6) follows from a structural result which classifies abstract Cohomological Field theories based on semi-simple Frobenius algebras (§3 below). The essential and difficult ingredient of the proof is the *Mumford conjecture*, proved by Madsen and Weiss [MW].

(2.8) When does the theorem apply? Semi-simplicity of quantum cohomology is a very strong restriction. Here are the main examples.

- (i) Projective spaces, Grassmannians, generalised flag varieties of GL .
- (ii) Projective toric varieties always have semi-simple *big* quantum cohomology.
- (iii) A method used by Givental in the toric case applies to varieties which have a circle action with isolated fixed-points: the circle-equivariant cohomology then gives a semi-simple deformation of the classical (and hence also the quantum) cohomology ring.
- (iv) Bayer [B] showed that semi-simplicity is preserved by blowing up points; in particular, there exist non-Fano examples.
- (v) 36 of the 59 families of 3D Fanos with no odd cohomology have been checked [AM, Ci].
- (vi) The abstract version of Theorem (2.6) applies to CohFT’s constructed from *Landau-Ginzburg B-models* for potentials with isolated critical points. This has not been sufficiently exploited; see open questions, below.
- (vii) On the negative side, if the *even* part of quantum cohomology is semi-simple, then the manifold has no odd cohomology, and, in the algebraic case, all cohomology is of type (p, p) [HMT]. This contradicts claims in the literature about some complete intersection Fanos.

(2.9) Dubrovin’s conjecture. A remarkable conjecture has gathered strong experimental support. Recall that an ordered collection $\{\mathcal{E}_i\}$ of objects in a triangulated \mathbb{C} -linear category is *exceptional* if $\text{Ext}^*(\mathcal{E}_i, \mathcal{E}_i) = \mathbb{C}$, concentrated in degree 0, while $\text{Ext}^k(\mathcal{E}_j, \mathcal{E}_i) = 0, \forall k$, when $j > i$. The collection is *complete* if it generates the (triangulated) category.

2.10 Conjecture (Dubrovin). *A projective manifold has generically semi-simple quantum cohomology iff its derived category of coherent sheaves contains a complete exceptional collection.*

2.11 Remark. Dubrovin also relates the Euler characteristics of the Ext-groups to quantum cohomology data.

For instance, the conjecture is known for rational surfaces [D1, B]. Ciulli [Ci] also checks this for 36 families of 3-dimensional Fanos. The conjecture is also confirmed for projective toric manifolds, by Kawamata’s theorem [K] on the existence of exceptional objects and knowledge of their quantum cohomology.

The conjecture would follow from sufficiently optimistic formulations of Mirror symmetry: the mirror partner for Gromov-Witten theory for a manifold X as in the conjecture would be a *Landau-Ginzburg B-model* whose potential had isolated Morse singularities (at generic values of the deformation space). But then, its Fukaya-Seidel category should contain a complete exceptional collection, which by the opposite mirror symmetry would give the desired exceptional objects in the derived category of X . This method has been essentially confirmed for projective toric manifolds by Abouzaid [Ab], who proves the required mirror symmetry (B -side on X is equivalent to A -side on the Landau-Ginzburg model) *without a priori* reference to exceptional objects (and, in particular, recovers Kawamata’s theorem).

(2.12) *Related results of Kontsevich.* In the 1990’s, Kontsevich initiated a programme, *Homological Mirror Symmetry*, which should give a far-reaching adaptation of Givental’s reconstruction conjecture without the semi-simplicity assumption. (The programme preceded Givental’s cited work, but converged with it later.) For a recent update, see [KKP].

A key step is to replace the notion of cohomological field theory with that of *chain-level, open-closed field* theory. Costello’s work [C] gives an implementation of these ideas. This more sophisticated approach seems necessitated by the fact that cohomological field theories seem unclassifiable with our limited understanding of Deligne-Mumford spaces. The semi-simple classification was a pleasant surprise.

Applying this programme to Gromov-Witten theory requires the construction a good *Fukaya category* for compact symplectic manifolds. This has not yet been accomplished in general. On the other hand, Kontsevich’s programme does appear to settle the partner side, *B-model* of mirror symmetry, where the analogue of the Fukaya category, the derived category of coherent sheaves, is much better understood.

3. Cohomological Field Theory

A Cohomological Field theory generalises the notion of a Frobenius algebra to the situation where surfaces vary in families, and the values of the theory are (matrices in) the cohomology classes in the base space of the family, instead of numbers. CohFT’s were introduced by Kontsevich and Manin [KM] precisely with Gromov-Witten theory in mind, although related notions (such as Segal’s Topological Conformal Field theory, taken up by Costello and Kontsevich) had an independent development.

I refer to [KM] for precise definitions and to the account in [T] of several possible variations, but here are the salient points. We start with a Frobenius algebra A , and require that:

- A family of closed surfaces over a base B gives a class in $H^*(B; \mathbb{C})$.
- With m input and n output boundaries, we get a class in $H^*(B; \text{Hom}(A^{\otimes m}; A^{\otimes n}))$.
- “Gluing boundaries = composition” applies in families.
- Nodal degenerations of surfaces (Lefschetz fibrations) are permitted.
- Everything is functorial in the base B .
- To make contact with the notion of [KM], all surfaces are assumed to be stable.
- The *flat vacuum* condition asks that “inserting a vacuum state should do nothing”; see (3.1) below.
- Finally, there is a homogeneity condition with respect to a specified *Euler vector field* on the Frobenius manifold of the CohFT (see §2.2); I will not spell out here.

Functoriality ensures that it suffices to specify the classes for the universal Lefschetz fibrations of stable surfaces, namely the universal curves \overline{C}_g^n over the *Deligne-Mumford spaces* \overline{M}_g^n of *stable curves* of genus g with n marked points. The collection of these classes is then subject to a set of constraints, which can be concisely formulated by stating that A is an algebra over the homology operad of the collection of Deligne-Mumford spaces.¹

More precisely, to each pair (n, g) describing the genus and number of marked points on a stable pointed curve, we must assign a class $\overline{Z}_g^n \in H^*(\overline{M}_g^n; A^{\otimes n})$. Restriction to boundary divisors is subject to *factorisation rule*. Thus, on a boundary stratum $\overline{M}_{g'}^{n'} \times \overline{M}_{g''}^{n''}$, corresponding to a splitting of the curve into two components of genera g', g'' joined at a node, the restricted \overline{Z}_g^n is described in terms of $\overline{Z}_{g'}^{n'} \wedge \overline{Z}_{g''}^{n''}$: namely, the latter product must be contracted with the Frobenius bilinear form applied to the two arguments corresponding to the nodes (one in each factor). A similar constraint arises at boundary strata corresponding to irreducible nodal curves. The flat vacuum condition demands that

$$1 \dashv \overline{Z}_g^{n+1} = f_1^* \overline{Z}_g^n, \quad (3.1)$$

where $f_1 : \overline{M}_g^{n+1} \rightarrow \overline{M}_g^n$ is the morphism forgetting the first marked point, and $1 \dashv$ stands for contraction with the “vacuum”, the identity $1 \in A$. (This is the only sensible implementation of the “vacuum does nothing” command, as we need an extra point to insert the vacuum vector!) The homogeneity constraint pertains to the classes \overline{Z}_g^n , when viewed as tensors on the Frobenius manifold of the CohFT [T, §7].

3.2 Theorem ([T]). *A CohFT based on a semi-simple Frobenius algebra A is determined by a power series $R(z) \in \text{End}(A)\llbracket z \rrbracket$, $R = \text{Id} + O(z)$, which is subject to Givental’s symplectic constraint $R(z)R^*(-z) \equiv \text{Id}$.*

The additional homogeneity constraint enforces a restriction on R (its k th Taylor coefficient R_k has degree $(-k)$, as a tensor on the Frobenius manifold). This constraint turns out to determine R uniquely. For example, in Gromov-Witten theory, A is the quantum cohomology $QH^*(X)$ at some semi-simple point u of the Frobenius manifold. The Taylor coefficients of R are then recursively determined from $R_0 = \text{Id}$ by the equations²

$$[R_{k+1}, (v \cdot)] = (\mu + k) \circ R_k$$

where $(v \cdot)$ is quantum multiplication by the Euler vector v of Remark 2.3, and μ is the linear operator $(\deg/2 - \dim_{\mathbb{C}} X \cdot \text{Id})$ on $A = H^*(X)$.

3.3 Remark. The Taylor series R can be interpreted in relation to *Dubrovin’s monodromy data* description of the Frobenius manifold. Namely, the $\text{End}(A)$ -valued expression

$$F(z) := R(1/z) \circ \exp(z(v \cdot)) \quad (3.4)$$

describes the asymptotics around $z = \infty$ of solutions to the *quantum differential equation*

$$\frac{\partial F}{\partial z} = \left((v \cdot) + \frac{\mu}{z} \right) F. \quad (3.5)$$

In other words, in suitable sectors centred at $z = \infty$ and for suitable solutions F , $R(1/z)$ gives the asymptotic expansion of the function $F(z) \circ \exp(-z(v \cdot))$. In the case of quantum cohomology, a distinguished solution to this equation can be described from the *1-point (J-)function*, [D2, Example 2.3]. (This is essentially the calibration of the Frobenius manifold, mentioned in §2.2.) The genuine solutions to (3.5) are multi-valued, with monodromy $z^{c_1(X) \cup}$ (classical cup-product); so the expression for F in (3.4), which is formally single-valued, cannot be a genuine function, and so R is not convergent.

¹We must regard this as a modular operad.

²When $(v \cdot)$ has repeated eigenvalues, solvability imposes constraints on μ .

(3.6) *Moral Interpretation of R .* The Frobenius algebra A is associated to the circle in a 2D field theory. Heuristically, it should be viewed as the cohomology of a space Y with circle action. In known applications, A is the Hochschild cohomology of a category (the category of boundary states in the open-closed theory), and so its chain-level model has an algebraic circle action. The series R is intended to give a *splitting* of the S^1 -equivariant cohomology:

$$H_{S^1}^*(Y) \cong H^*(Y) \otimes \mathbb{C}[[z]], \quad (3.7)$$

where z is the generator of $H_{S^1}^*(\text{point})$. Equivalently, this singles out the space of primary fields, a copy of A , inside the circle-equivariant version of A : the latter vector space is where the states to be inserted at points on the surface genuinely live.

The *existence* of such a splitting is *necessary* if one is to extend a cohomological field theory from the open stratum M_g^n of Deligne-Mumford moduli space over its boundary. This is not difficult to see, as follows. Consider for simplicity the setting of a space Y , as above. Boundaries of Deligne-Mumford space arise by degenerating handles on the curve to nodes. The cylinder with one incoming end and one outgoing end describes the identity map on $H_{S^1}^*(Y)$. The Deligne-Mumford degeneration of this cylinder into two crossing disks must extend this to a map

$$H_{S^1 \times S^1}^*(Y) \rightarrow H_{S^1 \times S^1}^*(Y),$$

but with the peculiar feature that, on the left, $S^1 \times S^1$ acts on Y via the first factor only, while the right, it acts only via the second factor. This is because the incoming and outgoing copies of Y are attached to the two different disks! The reader can now check that the existence of such an extension forces the existence of a splitting as in (3.7). A similar argument applies in a chain-level theory, enforcing a homotopy trivialisation of the circle action.

The core of the classification is the statement that a choice of splitting *determines* this boundary extension uniquely at the level of cohomology classes, in a semi-simple theory.

(3.8) *Construction of the theory.* The theory based on a given R can be constructed using the Morita-Mumford-Miller (tautological) classes and the so-called boundary classes on \overline{M}_g^n ; here, I just give a flavour of the universal formula and refer to [T, §5] for more details. Recall first that the κ -classes are integrals, along fibres of the universal surface bundle $\overline{C}_g^n \rightarrow \overline{M}_g^n$, of powers of the relative Euler class, and therefore behave *additively* under gluing of surfaces.

Start with the multiplicative class $\exp(\sum_j a_j \kappa_j) \in H^*(\overline{M}_g^n; A)$. Co-multiply this expression out to $A^{\otimes n}$, where one factor is attached to each of the n marked points. (It turns out that the coefficients $a_j \in A$ are pinned down from R and the flat vacuum condition.) Twist now each output factor by $R(\psi_i)$, with the ψ -class at the respective marked point.

As it stands, the class obtained turns out to violate the factorisation rules on Deligne-Mumford boundaries, and this must be corrected by the addition of boundary classes as follows. On each boundary divisor of \overline{M}_g^n , note the two Chern classes ψ_{\pm} of the cotangent lines at the node of the curve. Also note the class $\overline{Z}_{g'}^{n'} \wedge \overline{Z}_{g''}^{n''}$ coming from the lower-genus moduli spaces;³ the two factors are assumed to have been recursively corrected. We now contract the linear operator $(\text{Id} - R(\psi_+)^* R(\psi_-)) / (\psi_+ + \psi_-)$ with $\overline{Z}_{g'}^{n'} \wedge \overline{Z}_{g''}^{n''}$, at the two factors of A attached to the node. We push this forward to \overline{M}_g^n using Thom class of the boundary divisor, to produce the boundary correction term for \overline{Z}_g^n .

This construction can be captured more concisely by defining an action of the matrices $R(z)$ on the cohomology of Deligne-Mumford spaces. This action lifts Givental's loop group action on Gromov-Witten potentials, defined in his framework of quantised quadratic Hamiltonians [G2].

³Or its counterpart \overline{Z}_{g-1}^{n+2} on a boundary produced by a non-splitting handle of the curve.

(3.9) *What makes the classification work?* The *Euler class* of a Frobenius algebra A is the product of the co-product of 1: $1 \mapsto A \otimes A \mapsto A$. Pictorially, this is represented by a torus with one outgoing boundary. For the cohomology ring of a manifold, this is the usual Euler class, and it always squares to zero. However, the quantum Euler class *can* be *invertible*: this happens precisely when the quantum multiplication is semi-simple.

Hence, in the semi-simple case, one can increase the genus of surfaces without loss of information in the CohFT. Now, the *Mumford conjecture* (Madsen-Weiss) describes the complex cohomology of the open moduli space M_g^n of smooth curves in the $g \rightarrow \infty$ limit as a free \mathbb{C} -algebra in the *tautological* classes κ_j, ψ_i . From here, we can classify the restriction to M_g^n of semi-simple CohFT's.

Finally, it follows from a result of Looijenga's [L] that, in large g the boundary divisors of \overline{M}_g^n have Euler classes which are not zero-divisors. This controls the problem of extending cohomology classes to the boundary.

4. Some open questions

- *Reconstruction from classical data.* The reconstruction theorem requires some quantum cohomology information. A similar-looking result in classical singularity theory starts from more “classical” data. Namely, Saito's work shows how to produce Frobenius manifold structures on the unfolding space of a function f with an isolated critical point. The associated CohFT is the *Landau-Ginzburg B-model* of mirror symmetry. In this model, the analogue of $H^*(X)$ is the Jacobian ring of f , and in some ways the first Chern class is the residue class of f therein. The quantum cohomologies corresponds to the Jacobian ring of deformations of f . Now, a theorem of Scherk [Sc] asserts that, up to a change of coordinates in the ambient space, the singularity is determined from the Jacobian ring and the residue class of f . This does not quite determine the Frobenius manifold, but carries quite a bit of information (the *F-manifold*, missing the metric); the missing information has been completely described by Saito, as the choice of a *primitive volume form*.

Question: Is the quantum cohomology F -manifold determined by the classical $H^*(X)$ and $c_1(X)$, at least for varieties with generically semi-simple quantum cohomology? What minimal extra data is needed to recover the Frobenius manifold?

- *Degeneration.* Semi-simple theories come in families with non-semi-simple degenerations: classical cohomology for GW theory, or the Jacobian ring for the Landau-Ginzburg B -model potential with an isolated critical point. The Givental data for semi-simple theories degenerates at such a classical point. Nonetheless, some theories are continuous.

Problem: Understand this phenomenon; in particular, provide a formula for the higher-genus part of the Landau-Ginzburg B -model.

- *Formality.* Gromov-Witten theory can be defined at *chain level*, because the virtual fundamental classes are define from actual spaces with obstruction bundles. A cohomological classification leaves open the possibility of higher operations, in the style of Massey products in cohomology.

Question: Can we have higher operations in semi-simple field theories?

The expected answer is no. Formality of Deligne-Mumford spaces erases the difference between the chain operad and its homology. The proof of the classification theorem uses Euler classes to produce homological splittings, which appear to work at chain level. However, it is a bit more difficult than it seems to define the chain-level theory with all the trappings of a CohFT — mind, for instance, the fact that the quantum cup-product breaks the grading!

- *Twisted Frobenius manifolds.* Gromov-Witten K -theory and the twisted Gromov-Witten invariants (introduced by Coates and Givental) and other generalised cohomology examples

do not fit the standard definition of Frobenius manifolds. Variations of the notion have been studied by Dubrovin and Manin [M2].

Problem: Describe Givental's higher genus reconstruction in generalised cohomology.

This is not an idle generalisation, as K -theory seems particularly suited to equivariant calculations.

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