

Open-closed field theories, string topology, and Hochschild homology

Andrew Blumberg [*]	Ralph L. Cohen [†]	Constantin Teleman [‡]
Dept. of Mathematics	Dept. of Mathematics	Dept. of Mathematics
Stanford University	Stanford University	UC Berkeley
Stanford, CA 94305	Stanford, CA 94305	Berkeley, CA 94720

May 7, 2009

Abstract

In this expository paper we discuss a project regarding the string topology of a manifold, that was inspired by recent work of Moore-Segal [28] and Costello [17] on “open-closed topological conformal field theories”. In particular, given a closed, oriented manifold M , we describe the “string topology category” \mathcal{S}_M , which is enriched over chain complexes over a fixed field k . The objects of \mathcal{S}_M are closed, oriented submanifolds N of M , and the space of morphisms between N_1 and N_2 is a chain complex homotopy equivalent to the singular chains $C_*(\mathcal{P}_{N_1, N_2})$ where \mathcal{P}_{N_1, N_2} is the space paths in M that start in N_1 and end in N_2 . The composition pairing in this category is a chain model for the open string topology operations of Sullivan [35], and expanded upon by Harrelson [23] and Ramirez [31]. We will describe a calculation yielding that the Hochschild homology of the category \mathcal{S}_M is the homology of the free loop space, LM . Another part of the project is to calculate the Hochschild cohomology of the open string topology chain algebras $C_*(\mathcal{P}_{N, N})$ when M is simply connected, and relate the resulting calculation to $H_*(LM)$. These calculations generalize known results for the extreme cases of $N = \text{point}$ and $N = M$, in which case the resulting Hochschild cohomologies are both isomorphic to $H_*(LM)$. We also discuss a spectrum level analogue of the above results and calculations, as well as their relations to various Fukaya categories of the cotangent bundle T^*M with its canonical symplectic structure.

Contents

1 Open-closed Topological Field Theories

4

^{*}The first author was partially supported by an NSF postdoctoral fellowship

[†]The second author was partially supported by NSF grant DMS-0603713

[‡]The third author was partially supported by a research grant from the NSF

1.1	The work of Moore and Segal on open-closed TQFT's	6
1.2	The work of Costello on open-closed TCFT's	9
2	The string topology category and its Hochschild homology	12
3	Relations with the Fukaya category of the cotangent bundle	18

Introduction

In an open-closed topological field theory, one studies cobordisms between compact one-dimensional manifolds, whose boundary components are labeled by an indexing set, \mathcal{D} . The cobordisms are those of manifolds with boundary, that preserve the labeling sets in a specific way. The set of labels \mathcal{D} are referred to as “D-branes”, and in the string theory literature these are boundary values of “open strings”. An open-closed field theory is a monoidal functor from a category built out of such manifolds and cobordisms, that takes values in a linear category, such as vector spaces, chain complexes, or even the category of spectra. In this paper we will discuss two flavors of such open-closed field theories: “topological quantum field theories” (TQFT) as introduced by Moore and Segal [28], and “topological conformal field theories”, (TCFT), as studied by Getzler [21] and Costello [17].

The open part of such a theory \mathcal{F} is the restriction of \mathcal{F} to the “open subcategory”. This is the full subcategory generated by those compact one-manifolds, all of whose path components have nonempty boundary. As Moore and Segal originally pointed out, the data of an open field theory can be encoded in a category (or as Costello points out, an A_∞ -category when \mathcal{F} is an open-closed TCFT), $\mathcal{C}_\mathcal{F}$. The objects of $\mathcal{C}_\mathcal{F}$ are the set of D -branes, \mathcal{D} . The space of morphisms between λ_0 and $\lambda_1 \in \mathcal{D}$ is given by the value of the theory \mathcal{F} on the object I_{λ_0, λ_1} , defined by the interval $[0, 1]$ where the boundary component 0 is labeled by λ_0 , and 1 is labeled by λ_1 . We denote this vector space by $\mathcal{F}(\lambda_0, \lambda_1)$. The composition rules in this (A_∞) category are defined by the values of \mathcal{F} on certain “open-closed” cobordisms. Details of this construction will be given below.

In this paper we will report on a project whose goal is to understand how the “String Topology” theory of a manifold fits into this structure. This theory, as originally introduced by Chas and Sullivan [9] starts with a closed, oriented n -dimensional manifold M . It was shown in [11] that there is a (positive boundary) TQFT \mathcal{S}_M , which assigns to a circle the homology of the free loop space,

$$\mathcal{S}_M(S^1) = H_*(LM; k)$$

with field coefficients. This was recently extended by Godin [22] to show that string topology is actually an open-closed homological conformal field theory. In this theory the set of D-branes \mathcal{D} is the set of closed, oriented, connected submanifolds of M . The theory assigns to a compact one-manifold c with boundary levels, the homology of the mapping space,

$$\mathcal{S}_M(c) = H_*(Map(c, \partial; M)).$$

Here $Map(c, \partial; M)$ refers to the space of maps $c \rightarrow M$ that take the labeled boundary components to the submanifolds determined by the labeling. In particular, we write $\mathcal{P}_{N_0, N_1} = Map(I_{N_0, N_1}, \partial; M)$ for the space of paths $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) \in N_0$, and $\gamma(1) \in N_1$. In Godin's theory, given any open-closed cobordism Σ_{c_1, c_2} between one-manifolds c_1 and c_2 , there are homological operations

$$\mu_{\Sigma_{c_1, c_2}} : H_*(BDiff(\Sigma_{c_1, c_2}); k) \otimes H_*(Map(c_1, \partial; M); k) \longrightarrow H_*(Map(c_2, \partial; M); k).$$

An open-closed topological conformal field theory in the sense of Costello is a *chain complex* valued theory, and it is conjectured that the string topology theory has the structure of such a theory. The following theorem, which we report on in this paper, gives evidence for this conjecture.

Theorem 1. *1. There is an A_∞ category, which by abuse of notation we also call \mathcal{S}_M , which is enriched over chain complexes over an arbitrary field k , whose objects are $\mathcal{D}_M =$ connected, oriented submanifolds of M , and whose space of morphisms $Mor_{\mathcal{S}_M}(N_1, N_2)$ are chain homotopy equivalent to the singular chains, $C_*(\mathcal{P}_{N_1, N_2})$. Furthermore the compositions in this category reflect the open-closed string topology operations on the level of homology.*

2. The Hochschild homology of this A_∞ -category \mathcal{S}_M is the homology of the free loop space,

$$HH_*(\mathcal{S}_M) \cong H_*(LM; k).$$

Given any fixed submanifold N , the space of self-morphisms, $Mor_{\mathcal{S}_M}(N, N) \simeq C_*(\mathcal{P}_{N, N})$ is a differential graded algebra. Again, on the level of homology, this algebra structure is the string topology product introduced by Sullivan [35]. In this note we ask the following question, and report on its answer in a variety of special cases. (See Theorem 9 below.) Details will appear in [7].

Question: Let M be a simply connected, closed submanifold. For which connected, oriented, closed submanifolds $N \subset M$ is the Hochschild cohomology of $C_*(\mathcal{P}_{N, N})$ isomorphic to the homology of the free loop space,

$$HH^*(C_*(\mathcal{P}_{N, N}), C_*(\mathcal{P}_{N, N})) \cong H_*(LM)$$

as algebras? The algebra structure of the left hand side is given by cup product in Hochschild cohomology, and on the right hand side by the Chas-Sullivan product string topology product.

We observe that the two extreme cases of an affirmative answer to this question are known. For example, when N is a point, $\mathcal{P}_{N, N}$ is the based loop space, ΩM , and the statement that $HH_*(C_*(\Omega M), C_*(\Omega M)) \cong H_*(LM)$ was known in the 1980's by work of Burghelea, Goodwillie, and others. The Hochschild cohomology statement then follows from Poincare duality. Similarly, when $N = M$, then $\mathcal{P}_{N, N} \simeq M$, and the string topology algebra on $C_*(\mathcal{P}_{N, N})$ corresponds, via Poincare duality, to the cup product in $C^*(M)$. The fact that the Hochschild cohomology of $C^*(M)$ is isomorphic to $H_*(LM)$ follows from work of J. Jones in the 1980's, and the fact that the ring structure corresponds to the Chas-Sullivan product was proved in [12]. In this note we are able

to report on a calculation of $HH^*(C_*(\mathcal{P}_{N,N}), C_*(\mathcal{P}_{N,N}))$ which yields an affirmative answer to this question in many cases. (See Theorem 9 below.) These cases include when the inclusion map $N \hookrightarrow M$ is null homotopic. Thus

$$HH^*(C_*(\mathcal{P}_{N,N}), C_*(\mathcal{P}_{N,N})) \cong H_*(LS^n)$$

for every connected, oriented, closed submanifold of a sphere S^n . Other cases when one gets an affirmative answer to the above question include when the inclusion $N \hookrightarrow M$ is the inclusion of the fiber of a fibration $p : M \rightarrow B$, or more generally, when $N \hookrightarrow M$ can be factored as a sequence of embeddings, $N = N_0 \hookrightarrow N_1 \hookrightarrow \dots \hookrightarrow N_i \hookrightarrow N_{i+1} \dots \hookrightarrow N_k = M$ where each $N_i \subset N_{i+1}$ is the inclusion of a fiber of a fibration $p_{i+1} : N_{i+1} \rightarrow B_{i+1}$. Another case where the answer is affirmative is when $M = N \times N$, and $N \hookrightarrow M$ is the diagonal embedding. From this case we will be able to describe results and conjectures of the Hochschild cochain construction, $CH^* : DGA \rightarrow DGA$.

We point out that an amusing aspect of this question is that for any $N \hookrightarrow M$ for which the answer is affirmative, then one can use this submanifold as a single D -brane and recover $H_*(LM)$ as a Hochschild cohomology ring (i.e. “one brane is enough”), and that all such branes yield the same answer.

This paper is organized as follows. In section one below we discuss the relevant background of open-closed topological field theories, including a review of work of Moore and Segal [28], and of Costello [17]. In section 2 we describe the ingredients of the proofs of Theorems 1 and discuss the Hochschild cohomology calculations of the chain algebras, $C_*(\mathcal{P}_{N,N})$ (Theorem 9 below). The methods involve a type of Morita theory, and we also describe a consequence of these arguments, yielding that certain module categories over the algebras $C_*(\mathcal{P}_{N,N})$ are isomorphic. The extreme examples mentioned above yield isomorphisms of certain module categories over the cochains $C^*(M)$ and the chains of the based loop space, $C_*(\Omega M)$, which were originally obtained in [18]. See Theorem 10 below. In section 3 we discuss possible relationships between the categories described here and certain Fukaya categories of the cotangent bundle, T^*M with its canonical symplectic structure.

1 Open-closed Topological Field Theories

As mentioned in the introduction, the objects of study in an open-closed field theory are parameterized, compact, oriented one-manifolds, c , together with a labeling of the components of the boundary, ∂c , by elements of a set, \mathcal{D} . An “open-closed” cobordism Σ_{c_1, c_2} between two objects c_1 and c_2 is an oriented surface Σ , whose boundary is partitioned into three parts: the incoming boundary, $\partial_{in}\Sigma$ which is identified with c_1 , the outgoing boundary $\partial_{out}\Sigma$ which is identified with c_2 , and the remaining part of the boundary, referred to as the “free part”, $\partial_{free}\Sigma$ whose path components are labeled by \mathcal{D} , with the property that $\partial_{free}\Sigma$ is itself a cobordism between ∂c_1 and ∂c_2 , preserving the labeling. This is the usual notion of a cobordism of manifolds with boundary, with the additional data of the labeling set \mathcal{D} . Figure 1 below is a picture of a one-manifold whose boundary

components are labeled by elements of \mathcal{D} , and figure 2 is a picture of an open-closed cobordism. In this picture the free part of the boundary, $\partial_{free}\Sigma$ is highlighted in red. In figure 3 a smooth surface is shown that is homeomorphic to the open-closed cobordism given in figure 2. The free part of the boundary is again highlighted in red.

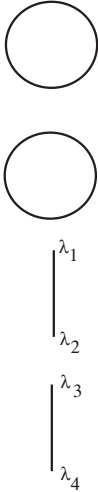


Figure 1: A one manifold with labels $\lambda_i \in \mathcal{D}$

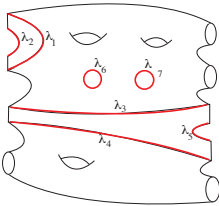


Figure 2: An open-closed cobordism

In [28], Moore and Segal describe basic properties of *open-closed topological quantum field theories*, and in a sense, Costello then gave a derived version of this theory when he gave a description of *open-closed topological conformal field theories*.

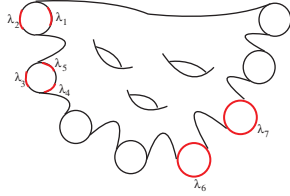


Figure 3: A smooth open-closed cobordism

1.1 The work of Moore and Segal on open-closed TQFT's

In [28] Moore and Segal describe how an *open-closed topological quantum field theory* \mathcal{F} assigns to each one manifold c with boundary components labeled by \mathcal{D} , a vector space over a field k , $\mathcal{F}(c)$. The theory \mathcal{F} also assigns to every diffeomorphism class of open closed cobordism Σ_{c_1, c_2} a linear map

$$\mathcal{F}(\Sigma_{c_1, c_2}) : \mathcal{F}(c_1) \rightarrow \mathcal{F}(c_2).$$

This assignment is required to satisfy two main properties:

1. 1. *Gluing*: One can glue two open-closed cobordisms when the outgoing boundary of one is identified with the incoming boundary of the other:

$$\Sigma_{c_1, c_2} \# \Sigma_{c_2, c_3} = \Sigma_{c_1, c_3}.$$

In this case the operation $\mathcal{F}(\Sigma_{c_1, c_2} \# \Sigma_{c_2, c_3})$ is required to be the composition:

$$\mathcal{F}(\Sigma_{c_1, c_2} \# \Sigma_{c_2, c_3}) : \mathcal{F}(c_1) \xrightarrow{\mathcal{F}(\Sigma_{c_1, c_2})} \mathcal{F}(c_2) \xrightarrow{\mathcal{F}(\Sigma_{c_2, c_3})} \mathcal{F}(c_3).$$

This condition can be viewed as saying that \mathcal{F} is a functor $\mathcal{F} : \mathcal{C}_{\mathcal{D}} \rightarrow Vect_k$, where $\mathcal{C}_{\mathcal{D}}$ is the cobordism category whose objects are one manifolds with boundary labels in \mathcal{D} , and whose morphisms are diffeomorphism classes of open-closed cobordisms. Here the diffeomorphisms are required to preserve the orientations, as well as the boundary structure (∂_{in} , ∂_{out} , and the labeling). $Vect_k$ is the category of vectors spaces over the field k , whose morphisms are linear transformations between them.

2. *Monoidal*: There are required to be natural isomorphisms,

$$\mathcal{F}(c_1) \otimes \mathcal{F}(c_2) \xrightarrow{\cong} \mathcal{F}(c_1 \sqcup c_2)$$

that makes \mathcal{F} into a monoidal functor. (The monoid structure in $\mathcal{C}_{\mathcal{D}}$ is given by disjoint union of both the object manifolds and the morphism cobordisms.)

Let \mathcal{F} be an open-closed TQFT. The value of \mathcal{F} on a single unit circle, $\mathcal{F}(S^1)$ is known as the *closed state space* of the theory \mathcal{F} . It is well known that \mathcal{F} is a commutative *Frobenius algebra* over k . That is, there is an associative multiplication $\mu_{\mathcal{F}} : \mathcal{F}(S^1) \otimes \mathcal{F}(S^1) \rightarrow \mathcal{F}(S^1)$ coming from the value of \mathcal{F} on the pair of pants cobordism from $S^1 \sqcup S^1$ to S^1 . The unit disk, viewed as having one outgoing boundary component, $\partial_{out}D^2 = S^1$, is a cobordism from the emptyset \emptyset to S^1 , and therefore induces a map $\iota : k \rightarrow \mathcal{F}(S^1)$, which is the unit in the algebra structure. Thinking of the disk as having one *incoming* boundary component, $\partial_{in}D^2 = S^1$, induces a map $\theta_{\mathcal{F}} : \mathcal{F}(S^1) \rightarrow k$ which is the “trace map” in the theory. That is, the bilinear form

$$\langle , \rangle : \mathcal{F}(S^1) \times \mathcal{F}(S^1) \xrightarrow{\mu_{\mathcal{F}}} \mathcal{F}(S^1) \xrightarrow{\theta_{\mathcal{F}}} k$$

is nondegenerate.

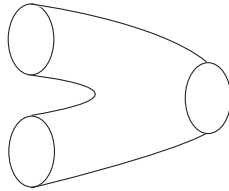


Figure 4: The pair of pants cobordism inducing the multiplication $\mu_{\mathcal{F}} : \mathcal{F}(S^1) \otimes \mathcal{F}(S^1) \rightarrow \mathcal{F}(S^1)$.

There is more algebraic structure associated to an open closed field theory \mathcal{F} . As described by Moore and Segal, there is a category, $\mathcal{C}_{\mathcal{F}}$ associated to the open part of the field theory.

Definition 2. *The category $\mathcal{C}_{\mathcal{F}}$ associated to an open-closed TQFT \mathcal{F} has as its objects the set of D -branes, \mathcal{D} . The space of morphisms between objects λ_1 and λ_2 is given by the value of the field theory \mathcal{F} on the one-manifold I_{λ_1, λ_2} which is given by the interval $[0, 1]$ with boundary components labeled by λ_1 and λ_2 . We write this space as $\mathcal{F}(\lambda_1, \lambda_2)$.*

The composition law in the category $\mathcal{C}_{\mathcal{F}}$ is defined by the open-closed cobordism shown in figure 6.

Notice that the endomorphism algebras in this category, $\mathcal{F}(\lambda, \lambda)$, are also Frobenius algebras. For simplicity we write these algebras as $\mathcal{F}(\lambda)$. The trace maps are induced by the open-closed cobordism between $I_{\lambda, \lambda}$ and the empty set given by the disk as in figure 7.

We observe that the closed state space $\mathcal{F}(S^1)$ is necessarily commutative as an algebra, because the cobordisms shown in figure 8 admit an orientation preserving diffeomorphism between them that fixes the boundary pointwise. However for a D -brane $\lambda \in \mathcal{D}$, the fact that the open-closed cobordisms shown in figure 9 are *not* diffeomorphic via an orientation preserving diffeomorphism



Figure 5: The one manifold I_{λ_1, λ_2} which induces the morphism space $\mathcal{F}(\lambda_1, \lambda_2)$.

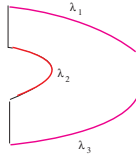


Figure 6: The value of \mathcal{F} on this cobordism defines the composition pairing, $\mathcal{F}(\lambda_1, \lambda_2) \otimes \mathcal{F}(\lambda_2, \lambda_3) \rightarrow \mathcal{F}(\lambda_1, \lambda_3)$

that fixes the incoming and outgoing boundaries, imply that the Frobenius algebra $\mathcal{F}(\lambda, \lambda)$ may not be commutative.

These algebras are, of course, related to each other. For example, the “whistle” open-closed cobordism given in figure 10 defines a ring homomorphism $\theta_\lambda : \mathcal{F}(\lambda) \rightarrow \mathcal{F}(S^1)$, which, is easy to see takes values in the center $Z(\mathcal{F}(\lambda))$ (see [28] for details.) So in particular one has the following result.

Proposition 3. *Any open-closed TQFT \mathcal{F} comes equipped with map of algebras*

$$\theta_\lambda : \mathcal{F}(S^1) \rightarrow Z(\mathcal{F}(\lambda))$$

where $Z(\mathcal{F}(\lambda))$ is the center of the endomorphism algebra $\mathcal{F}(\lambda)$, for any $\lambda \in \mathcal{D}$.



Figure 7: The value of \mathcal{F} on this open-closed cobordism defines the trace map $\mathcal{F}(\lambda) \rightarrow k$

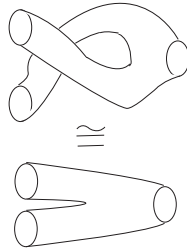


Figure 8: These diffeomorphic cobordisms imply that the Frobenius algebra $\mathcal{F}(S^1)$ is commutative.

Turning the whistle cobordism around, so that its incoming boundary is $I_{\lambda, \lambda}$, and its outgoing boundary is S^1 , defines a homomorphism $\theta_\lambda^* : \mathcal{F}(S^1) \rightarrow \mathcal{F}(\lambda)$, which is not difficult to see is adjoint to θ_λ , with respect to the inner products defined by the corresponding Frobenius algebras. Moreover, studying the relevant glued cobordisms, one can show that the composition, $\theta_\lambda \circ \theta_\lambda^*$ satisfies the “Cardy formula”,

$$\theta_\lambda \circ \theta_\lambda^*(\phi) = \sum_{i=1}^n \psi^i \phi \psi_i \quad (1)$$

where $\{\psi_1, \dots, \psi_n\}$ is any basis of $\mathcal{F}(\lambda)$, and $\{\psi^1, \dots, \psi^n\}$ is the dual basis (with respect to the inner product in the Frobenius algebra structure). Again, see [28] for the details of this claim.

1.2 The work of Costello on open-closed TCFT’s

In [17] Costello studied open-closed *topological conformal field theories* (TCFT). Such a theory can be viewed as a derived version of a topological quantum field theory, and in a sense, Costello’s work can, in part be viewed as a derived extension and generalization of the work of Moore and Segal.

More precisely, the TCFT’s Costello studied are functors,

$$\mathcal{F} : \mathcal{OC}_{\mathcal{D}} \rightarrow \text{Comp}_k$$

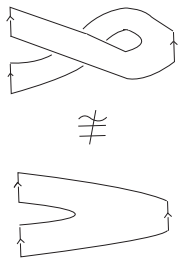


Figure 9: These surfaces are not diffeomorphic as open-closed cobordisms, and thus the Frobenius algebras $\mathcal{F}(\lambda)$ may not be commutative.



Figure 10: The “whistle open-closed cobordism” inducing the map $\theta_\lambda : \mathcal{F}(S^1) \rightarrow Z(\mathcal{F}(\lambda))$.

where $\mathcal{OC}_{\mathcal{D}}$ is an open-closed cobordism category, enriched over chain complexes, and $Comp_k$ is the symmetric monoidal category of chain-complexes over a ground field k . In Costello’s work, $Char(k) = 0$. By an “open-closed cobordism category, enriched over chain complexes”, Costello means the following. Let \mathcal{D} be an indexing set of “D-branes” as above. Then the objects of $\mathcal{OC}_{\mathcal{D}}$ are parameterized, compact, oriented one-manifolds, c , together with a labeling of the components of the boundary, ∂c , by elements of \mathcal{D} , as described in the previous section.

To describe the chain complex of morphisms between objects c_1 and c_2 , one considers the moduli space of all Riemann surfaces that form open-closed cobordisms between c_1 and c_2 . This moduli space was originally described by Segal [33] when the c_i ’s have no boundary. For the general situation we refer the reader to Costello’s paper [17]. These open-closed cobordisms are required to satisfy the additional “positive boundary” requirement, that every path component of an element $\Sigma \in \mathcal{M}_{\mathcal{D}}(c_1, c_2)$ has a nonempty incoming boundary. It is standard to see that

$$\mathcal{M}_{\mathcal{D}}(c_1, c_2) \simeq \coprod BDiff^+(\Sigma, \partial\Sigma)$$

where the disjoint union is taken over all diffeomorphism classes of open-closed cobordisms from c_1 to c_2 . These diffeomorphisms are diffeomorphisms of open-closed cobordisms, as defined in the previous section. They are orientation preserving, they preserve the incoming and outgoing boundaries pointwise, and they preserve the labelings in \mathcal{D} . The morphisms in $\mathcal{OC}_{\mathcal{D}}$ are then the singular chains with coefficients in k , $Mor_{\mathcal{OC}_{\mathcal{D}}}(c_1, c_2) = C_*(\mathcal{M}_{\mathcal{D}}(c_1, c_2); k)$.

A topological conformal field theory is then a functor $\mathcal{F} : \mathcal{OC}_{\mathcal{D}} \rightarrow \text{Comp}_k$ which is “ h -monoidal”, in the sense that there are natural transformations $\mathcal{F}(c_1) \otimes \mathcal{F}(c_2) \rightarrow \mathcal{F}(c_1 \sqcup c_2)$ which are quasi-isomorphisms of chain complexes. Costello calls $\mathcal{OC}_{\mathcal{D}} - \text{mod}$ the functor category of topological conformal field theories.

Let $\mathcal{O}_{\mathcal{D}} \hookrightarrow \mathcal{OC}_{\mathcal{D}}$ be the full subcategory whose objects have no closed components. That is, every connected component of a one-manifold $c \in \text{Ob}(\mathcal{O}_{\mathcal{D}})$ has (labeled) boundary. Write $\mathcal{O}_{\mathcal{D}} - \text{mod}$ to be the functor category of h -monoidal functors $\phi : \mathcal{O}_{\mathcal{D}} \rightarrow \text{Comp}_k$. We refer to such a functor as an “open-field theory”.

Costello observed that an open topological conformal field theory $\phi : \mathcal{O}_{\mathcal{D}} \rightarrow \text{Comp}_k$ defines an A_{∞} -category, enriched over chain complexes, in much the same way as an open topological quantum field theory defines a category (see Definition (2) above). This is most easily seen if the field theory is *strictly* monoidal, that is, the transformations $\phi(c_1) \otimes \phi(c_2) \rightarrow \phi(c_1 \sqcup c_2)$ are *isomorphisms* of chain complexes, rather than only quasi-isomorphisms. In this case the associated A_{∞} -category, which we call \mathcal{C}_{ϕ} , has objects given by the set of D -branes \mathcal{D} . The space of morphisms $\phi(\lambda_0, \lambda_1)$ is the chain complex given by the value of the functor ϕ on the object I_{λ_0, λ_1} . We call this space $\phi(\lambda_0, \lambda_1)$. The higher compositions

$$\phi(\lambda_1, \lambda_2) \otimes \phi(\lambda_2, \lambda_3) \otimes \cdots \otimes \phi(\lambda_{n-1}, \lambda_n) \longrightarrow \phi(\lambda_1, \lambda_n)$$

are given by the value of the functor ϕ on the open-closed cobordism between $\coprod_{i=1}^{n-1} I_{\lambda_i, \lambda_{i+1}}$ and I_{λ_1, λ_n} given by the connected, genus zero surface $D_{\lambda_1, \dots, \lambda_n}$ pictured in figure 11 in the case $n = 4$.

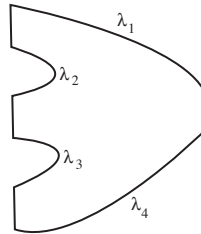


Figure 11: The open-closed cobordism $D_{\lambda_1, \dots, \lambda_4}$

The A_{∞} -category defined by an open TCFT has additional properties that Costello referred to as a “Calabi-Yau” A_{∞} category. The following theorem of Costello describes the central nature of this category in open-closed field theory.

Theorem 4. (Costello [17]) *a. The restriction functor $\rho : \mathcal{OC}_{\mathcal{D}}\text{-mod} \rightarrow \mathcal{O}_{\mathcal{D}}\text{-mod}$ from open-closed TCFT's to open TCFT's has a derived left adjoint, $L_{\rho} : \mathcal{O}_{\mathcal{D}}\text{-mod} \rightarrow \mathcal{OC}_{\mathcal{D}}\text{-mod}$.*

b. If $\phi \in \mathcal{O}_{\mathcal{D}}\text{-mod}$ is an open TCFT, then the closed state space of the open-closed field theory $L_{\rho}(\phi)$ (i.e the value of the functor on the object given by the circle, $L_{\rho}(\phi)(S^1)$) is a chain complex whose homology is given by the Hochschild homology of the A_{∞} -category, ϕ . That is,

$$H_*(L_{\rho}(\phi)(S^1)) \cong HH_*(\mathcal{C}_{\phi}).$$

Here the Hochschild homology of a category enriched over chain complexes is computed via the Hochschild complex, whose n -simplices are direct sums of terms of the form $Mor(\lambda_0, \lambda_1) \otimes Mor(\lambda_1, \lambda_2) \otimes \cdots \otimes Mor(\lambda_{n-1}, \lambda_n) \otimes Mor(\lambda_n, \lambda_0)$. This is a double complex whose boundary homomorphisms are the sum of the internal boundary maps in the chain complex of n -simplices, plus the Hochschild boundary homomorphism, which is defined as the alternating sum $\sum_{i=0}^n (-1)^i \partial_i$, where for $i = 0, \dots, n-1$, ∂_i is induced by the composition $Mor(\lambda_i, \lambda_{i+1}) \otimes Mor(\lambda_{i+1}, \lambda_{i+2}) \rightarrow Mor(\lambda_i, \lambda_{i+2})$. ∂_n is induced by the composition $Mor(\lambda_n, \lambda_0) \otimes Mor(\lambda_0, \lambda_1) \rightarrow Mor(\lambda_n, \lambda_1)$. The Hochschild homology of an A_{∞} -category enriched over chain complexes is defined similarly. See [17] for details.

Costello's theorem can be interpreted as saying that there is a "universal" open-closed theory with a given value on the open cobordism category (i.e the value of the derived left adjoint L_{ρ}), and that its closed state space has homology equal to the Hochschild homology of the associated A_{∞} -category. We note that in the interesting case when there is only one \mathcal{D} -brane, that is, $\mathcal{D} = \{\lambda\}$, then the A_{∞} -category is an A_{∞} -algebra, and so the closed state space of the associated universal open-closed theory would have homology given by the Hochschild homology of this algebra. In particular this says that for any open-closed field theory ϕ with one \mathcal{D} -brane, which has the corresponding A_{∞} -algebra A , then there is a well defined map from the Hochschild homology $HH_*(A) \rightarrow H_*(\phi(S^1))$. This can be viewed as a derived version of the Moore-Segal result (Proposition 3) that gives a map $\phi(S^1) \rightarrow Z(A)$. In the Moore-Segal setting, $\phi(S^1)$ is an ungraded Frobenius algebra, (or equivalently it has trivial grading) so we may identify it with $H_0(\phi(S^1))$. Furthermore the center $Z(A)$ may be identified with the zero dimensional Hochschild cohomology $HH^0(A)$, so that the Moore-Segal result gives a map $H_0(\phi(S^1)) \rightarrow HH^0(A)$. By the self-duality of the Frobenius algebra structures of $\phi(S^1)$ and of A , this gives a dual map $HH_0(A) \rightarrow H_0(\phi(S^1))$. Costello's map can be viewed as a derived version of this map.

2 The string topology category and its Hochschild homology

Among the goals of the project reported on here, is to see how string topology fits into Costello's picture. The most basic operation in string topology is the loop product defined by Chas and Sullivan [9]:

$$\mu : H_p(LM) \otimes H_q(LM) \rightarrow H_{p+q-n}(M)$$

where M is a closed, oriented, n -dimensional manifold. Now let $B : H_q(LM) \rightarrow H_{q+1}(LM)$ be the operation induced by the rotation S^1 -action on LM , $r : S^1 \times LM \rightarrow LM$.

$$\begin{aligned} B : H_q(LM) &\rightarrow H_{q+1}(S^1 \times LM) \xrightarrow{r_*} H_{q+1}(LM) \\ \alpha &\rightarrow r_*([S^1] \times \alpha) \end{aligned}$$

The following was one of the main theorems of [9]

Theorem 5. [9] *Let $\mathbb{H}_*(LM) = H_{*+n}(LM)$ be the (regraded) homology of the free loop space. Then with respect to the loop product μ and the degree one operator B , $\mathbb{H}_*(LM)$ has the structure of a (graded) Batalin-Vilkovisky algebra. That is, it is a graded commutative algebra satisfying the following identities:*

1. $B^2 = 0$, and
2. For $\alpha \in \mathbb{H}_p(LM)$, and $\beta \in \mathbb{H}_q(LM)$ the bracket operation

$$\{\alpha, \beta\} = (-1)^{|\alpha|} B(\alpha \cdot \beta) - (-1)^{|\alpha|} B(\alpha) \cdot \beta - \alpha \cdot B(\beta)$$

is a derivation in each variable.

Moreover, a formal argument given in [9] implies that the operation $\{, \}$ satisfies the (graded) Jacobi identity, and hence gives $\mathbb{H}_*(LM)$ the structure of a graded Lie algebra.

The product is defined by considering the mapping space, $Map(P, M)$ where P is the pair of pants cobordism (figure (4)) between two circles and one circle. By restricting maps to the incoming and outgoing boundaries, one has a correspondence diagram

$$LM \xleftarrow{\rho_{out}} Map(P, M) \xrightarrow{\rho_{in}} LM \times LM. \quad (2)$$

By retracting the surface P to the homotopy equivalent figure 8 graph, one sees that one has a homotopy cartesian square,

$$\begin{array}{ccc} Map(P, M) & \xrightarrow{\rho_{in}} & LM \times LM \\ \downarrow & & \downarrow \\ M & \xrightarrow{\Delta} & M \times M \end{array}$$

where $\Delta : M \hookrightarrow M \times M$ is the diagonal embedding. This then allows the construction of an ‘‘umkehr map’’ $\rho_{in}^! : H_*(LM \times LM) \rightarrow H_{*-n}(Map(P, M))$. This map was defined on the chain level in [9], and via a Pontrjain-Thom map $LM \times LM \rightarrow Map(P, M)^{TM}$ in [12]. Here $Map(P, M)^{TM}$ is the Thom space of the tangent bundle TM pulled back over the mapping space via evaluation at a basepoint, $Map(P, M) \rightarrow M$. By twisting with the virtual bundle $-TM$, Cohen and Jones proved the following.

Theorem 6. [12] *For any closed manifold M , the Thom spectrum LM^{-TM} is a ring spectrum. When given an orientation of M , the ring structure of LM^{-TM} induces, via the Thom isomorphism, the Chas-Sullivan algebra structure on $\mathbb{H}_*(LM)$.*

The Chas-Sullivan product was generalized to a TQFT by Cohen and Godin in [11]. Given a cobordism Σ between p -circles and q -circles, they considered the following correspondence diagram analogous to (2).

$$(LM)^q \xleftarrow{\rho_{out}} \text{Map}(\Sigma, M) \xrightarrow{\rho_{in}} (LM)^p. \quad (3)$$

Using fat (ribbon) graphs to model surfaces, Cohen and Godin described an umkehr map

$$\rho_{in}^! : H_*((LM)^p) \rightarrow H_{*+\chi(\Sigma)\cdot n}(\text{Map}(\Sigma, M))$$

which allowed the definition of an operation

$$\mu_\Sigma = (\rho_{out})_* \circ \rho_{in}^! : H_*((LM)^p) \rightarrow H_{*+\chi(\Sigma)\cdot n}((LM)^q)$$

which yielded the (closed) TQFT structure. In these formulae, $\chi(\Sigma)$ is the Euler characteristic of the cobordism Σ .

Open-closed operations were first defined by Sullivan in [35]. Somewhat later, Ramirez [31] and Harrelson [23] showed that these operations define a positive boundary, open-closed topological quantum field theory, in the Moore-Segal sense, except that the value of the theory lie in the category of graded vector spaces over a field k . In this theory, which we call \mathcal{S}_M , the closed state space is given by

$$\mathcal{S}_M(S^1) = H_*(LM; k). \quad (4)$$

The set of \mathcal{D} -branes \mathcal{D}_M is defined to be the set of connected, closed submanifolds $N \subset M$. The value of this theory on the interval labeled by submanifolds N_1 and N_2 (see figure (5)) is given by

$$\mathcal{S}_M(N_1, N_2) = \mathcal{S}_M(I_{N_1, N_2}) = H_*(\mathcal{P}_{N_1, N_2}), \quad (5)$$

where \mathcal{P}_{N_1, N_2} is the space of paths $\alpha : [0, 1] \rightarrow M$ with boundary conditions, $\alpha(0) \in N_1$, $\alpha(1) \in N_2$.

Finally, using families of ribbon graphs modeling both closed and open-closed cobordisms, in [22] Godin recently proved the following result.

Theorem 7. (Godin) [22] *Let $\mathcal{OC}_{\mathcal{D}_M}^{H^*}$ be the category with the same objects as $\mathcal{OC}_{\mathcal{D}_M}$, and whose morphisms are the homology of the morphisms in $\mathcal{OC}_{\mathcal{D}_M}$. That is, given objects c_1 and c_2 , the morphisms from c_1 to c_2 are given by*

$$\text{Mor}_{\mathcal{OC}_{\mathcal{D}_M}^{H^*}}(c_1, c_2) = H_*(\mathcal{M}_{\mathcal{D}}(c_1, c_2); k) \cong \bigoplus H_*(\text{BDiff}^+(\Sigma, \partial\Sigma); k)$$

where the disjoint union is taken over all diffeomorphism classes of open-closed cobordisms from c_1 to c_2 . Then the above string topology operations can be extended to a symmetric monoidal functor

$$\mathcal{S}_M : \mathcal{OC}_{\mathcal{D}_M}^{H^*} \rightarrow \text{GrVect}$$

where GrVect is the category of graded vector spaces over k , whose monoidal structure is given by (graded) tensor product. In other words, the string topology of M is a positive boundary, open-closed “homological conformal field theory” (HCFT).

Notice that being a homological conformal field theory is a weaker property than being a topological conformal field theory, and so Costello's results cannot be immediately applied to the string topology of a manifold M . In order for the functor \mathcal{S}_M to actually induce a TCFT, the string topology operations must be defined on the chain level, and satisfy the appropriate compatibility and coherence properties. It is conjectured that in fact this can be done. In any case, Costello's theorem (Theorem 4 above) suggests that there is an A_∞ -category associated to the string topology of M , and that its Hochschild homology should be the value of the closed state space, $\mathcal{S}_M(S^1) = H_*(LM; k)$. Theorem 1 as stated in the introduction, states that such a category exists. Its proof will appear in [7].

Another interesting question deals with when there is only one D -brane $\mathcal{D} = \{N\}$, where N is a fixed, connected submanifold of M . Then the corresponding A_∞ -category would be an A_∞ -algebra. It turns out that for Poincare duality reasons the correct Hochschild theory to use in this situation is Hochschild cohomology. The question described in the introduction, regarding the relationship between these Hochschild cohomology algebras and the Chas-Sullivan algebra structure on $H_*(LM)$, was based on the idea that string topology, even in this "one D -brane" setting should fit into Costello's picture of a universal open-closed TCFT. In particular the calculations reported on below state that for a large class of submanifolds $N \subset M$, the subcategory with whose only D -brane is N still yields the full closed state space of string topology,

$$HH^*(C_*(\mathcal{P}_{N,N}), C_*(\mathcal{P}_{N,N})) \cong H_*(LM) = \mathcal{S}_M(S^1).$$

The proofs of these statements involve the following ideas. By an *Eilenberg – Moore* argument, there is a chain homotopy equivalence $C_*(\mathcal{P}_{N_1, N_2}) \simeq C_*(F_{N_1}) \otimes_{C_*(\Omega M)}^L C_*(F_{N_1})$ where F_{N_i} is an appropriate homotopy fiber of the inclusion $N_i \hookrightarrow M$ which comes equipped with an action of a topological group of the homotopy type of ΩM (e.g the Kan simplicial loop group). The homotopy orbit space of this action is equivalent to N . This tensor product is an appropriate derived tensor product (in the category of chain complexes).

Next one uses an appropriate notion of Poincare duality with twisted coefficients. Here, instead of the classical setting where one has coefficients given by modules over the group ring $\mathbb{Z}[\pi_1(M)]$, one needs a derived version of Poincare duality, that applies for modules over $C_*(\Omega M)$. The appropriate duality was worked out by Malm in [27], using the work of Dwyer-Greenlees-Iyengar [18] and Klein [24]. Using this, one obtains the following isomorphism. This is related to results of Klein in [25].

$$C_*(\mathcal{P}_{N_1, N_2}) \simeq C_*(F_{N_1}) \otimes_{C_*(\Omega M)}^L C_*(F_{N_1}) \simeq \mathit{Rhom}_{C_*(\Omega M)}(C_*(F_{N_1}), C_*(F_{N_2}))$$

Here Rhom denotes a derived homomorphism group. This chain equivalence allows for the definition of the category \mathcal{S}_M . The objects are closed, connected, oriented submanifolds, $N \subset M$, and the morphism space from N_1 to N_2 is the derived homomorphism space, $\mathit{Rhom}_{C_*(\Omega M)}(C_*(F_{N_1}), C_*(F_{N_2}))$. The Hochschild (co)homology statements in the above theorems are approached using "Morita duality arguments". This is, in a sense that can be made precise, dual to the more common Morita

equivalence arguments. For example to prove Theorem , the following basic principle is proven in [7]:

Theorem 8. *Let R and S be two differential graded algebras over a field k , and suppose there exist $R - S$ (differential graded) modules satisfying the following equivalences:*

$$Rhom_R(P, Q) \simeq S \quad \text{and} \quad Rhom_S(P, Q) \simeq R. \quad (6)$$

Then their Hochschild cohomologies are isomorphic,

$$HH^*(R, R) \cong HH^*(S, S).$$

Using this result, given a submanifold $N \subset M$, one considers

$$R \simeq C_*(\Omega M), \quad S \simeq C_*(\mathcal{P}_{N,N}) \simeq Rhom_{C_*(\Omega M)}(C_*(F_N), C_*(F_N)).$$

The $R - S$ modules are both given by $P = Q \simeq C_*(F_N)$. We already know that $Rhom_R(P, Q) \simeq S$, for any $N \subset M$, when M is simply connected. The following result was achieved in consultation with W.Dwyer:

Theorem 9. *Assume M is simply connected. Then for any $N \subset M$ in \mathcal{D} ,*

$$Rhom_{C_*(\mathcal{P}_{N,N})}(C_*(F_N), C_*(F_N)) \simeq \hat{C}_*(\Omega M)$$

Here $\hat{C}_(\Omega M)$ is the completion of $C_*(\Omega M)$ with respect to the homology theory h_*^N , defined on the derived category of $C_*(\Omega M)$ -modules given by*

$$h_*^N(P) = Tor_{C_*(\Omega N)}(k, P).$$

So a natural question is: For what submanifolds $N \subset M$ is $C_*(\Omega M)$ h_*^N -complete? This is relevant, because in the presence of this completeness, one would know that $HH^*(C_*(\mathcal{P}_{N,N}), C_*(\mathcal{P}_{N,N})) \cong HH^*(C_*(\Omega M), C_*(\Omega M)) \cong H_*(LM)$.

We remark that as of this date, we know that $C_*(\Omega M)$ is h_*^N -complete in the following cases:

1. The inclusion map $N \hookrightarrow M$ is null homotopic. This implies that $F_N \simeq \Omega M \times N$, and $\mathcal{P}_{N,N} \simeq \Omega M \times N \times N$.
2. The inclusion $N \hookrightarrow M$ is the inclusion of the fiber of a fibration $p : M \rightarrow B$. More generally there is a sequence of inclusions, $N \subset N_1 \subset N_2 \subset \dots \subset N_k = M$ where each $N_i \subset N_{i+1}$ is the inclusion of the fiber of a fibration $p_{i+1} : N_{i+1} \rightarrow B_{i+1}$.
3. If $M = N \times N$, and N is embedded as the diagonal inclusion $\Delta : N \subset N \times N$.

As is usual in Morita theory, these arguments actually prove that certain categories of modules are isomorphic. Namely, the following result will also appear in [7]. For $N \subset M$ as above, let E_M be the endomorphism algebra $E_N = Rhom_{C_*(\Omega M)}(C_*(F_N), C_*(F_N))$, which as noted above is chain equivalent to $C_*(\mathcal{P}_{N,N})$.

Theorem 10. *For any $N \subset M$ for which $C_*(\Omega M)$ is h_*^N -complete, then the thick subcategory of differential graded modules over $C_*(\Omega M)$ generated by $C_*(F_N)$, is equivalent to the thick subcategory of E_N -modules generated by E_N .*

Notice that in the case when $N = M$, $C_*(F_N) \simeq k$, and $E_N \simeq C^*(M)$, which by Poincare duality is chain equivalent to $C_*(M) \simeq C_*(\mathcal{P}_{M,M})$. In this case the theorem says that the thick subcategory of $C_*(\Omega M)$ -modules generated by k is equivalent to the thick subcategory of $C^*(M)$ -modules generated by $C^*(M)$. The result in this case was proved by Dwyer, Greenlees, and Iyengar in [18].

As an application of the above calculations, one can obtain a result about iterating the the Hochschild cochain construction. Namely, notice that if A is a differential graded algebra, then the Hochschild cochains $CH^*(A, A)$ also inherits the structure of a differential graded algebra, under the cup product operation. For ease of notation, we write $CH^*(A)$ to denote $CH^*(A, A)$. We can then iterate the construction, and denote by $CH_n^*(A)$ to denote the n -fold iterate,

$$CH_n^*(A) = CH^*(CH^*(CH^*(\dots(CH^*(A))\dots))).$$

Now observe that if we consider the diagonal embedding, $\Delta : M \hookrightarrow M \times M$, then the space of paths $\mathcal{P}_{M,M}(M \times M)$ is homotopy equivalent to the free loop space, LM . Furthermore, the homotopy fiber, $F_M \simeq \Omega M$. Thus the chain complex

$$C_*(LM) \simeq C_*(\mathcal{P}_{M,M}(M \times M)) \simeq \mathit{Rhom}_{C_*(\Omega M \times \Omega M)}(C_*(\Omega M); C_*(\Omega M))$$

gives a chain level algebra structure, which the authors show to induce the Chas-Sullivan structure on $H_*(LM)$ in homology. The following can be viewed as a generalization of results of [12], relating the Chas-Sullivan algebra to Hochschild cohomology:

Corollary 11. *Let M be simply connected. Then there is a quasi-isomorphism of differential graded algebras,*

$$CH_n^*(C^*M) \simeq (C_*(LM))^{\otimes 2^{n-1}}.$$

In homology, the algebra structure on the right hand side is the Chas-Sullivan string topology product on $H_(L(M^{\times 2^{n-1}}))$.*

Sketch of proof. One knows that $CH^*(M) \simeq C_*(LM) \simeq C_*(\mathcal{P}_{M,M}(M \times M))$. Conjecture is known in the case of the embedding, $\Delta : M \rightarrow M \times M$, so we have that

$$CH_2^*(C^*(M)) \simeq CH^*(C_*(\mathcal{P}_{M,M}(M \times M))) \simeq C_*(L(M \times M)) \simeq C_*(LM)^{\otimes 2}.$$

But the last chain algebra is equivalent to $C_*(\mathcal{P}_{M \times M, M \times M}(M^4))$, where $M \times M$ is embedded in M^4 by the diagonal, $\Delta : M \times M \rightarrow M^4$. The result follows from iterating this procedure. \square

Finally we remark that there are spectrum level analogues of the above theorems, and in particular Theorems 1 and as stated in the introduction.

Theorem 12. *There is a string topology category, enriched over spectra, which by abuse of notation we still refer to as \mathcal{S}_M , whose objects again are elements of \mathcal{D} . The morphisms between N_1 and N_2 have the homotopy type of $\mathcal{P}_{N_1, N_2}^{-TN_1}$, the Thom spectrum of the virtual bundle $-TN_1$, where TN_1 is the tangent bundle of N_1 , pulled back over \mathcal{P}_{N_1, N_2} via the evaluation map that takes a path $\alpha \in \mathcal{P}_{N_1, N_2}$ to its initial point $\alpha(0) \in N_1$. Then the Topological Hochschild Homology of this category is the suspension spectrum of the free loop space (with a disjoint basepoint),*

$$THH_\bullet(\mathcal{S}_M) \simeq \Sigma^\infty(LM_+). \quad (7)$$

Moreover, the analogue of Theorem 12 is the following:

Theorem 13. *Assume that M is a simply connected, closed manifold. Then given any closed, oriented submanifold $N \subset M$, then the Thom spectrum $\mathcal{P}_{N, N}^{-TN}$ is a ring spectrum, and its Topological Hochschild Cohomology is given by*

$$THH^\bullet(\mathcal{P}_{N, N}^{-TN}) \simeq LM^{-TM} \quad (8)$$

and the equivalence is one of ring spectra.

3 Relations with the Fukaya category of the cotangent bundle

This section is speculative, regarding the possible relationships between the string topology category \mathcal{S}_M , and the Fukaya category of the cotangent bundle T^*M . The Fukaya category is an A_∞ -category associated to a symplectic manifold (N^{2n}, ω) . Here $\omega \in \Omega^2(N)$ is a symplectic 2-form. Recall that for any smooth n -manifold M^n , T^*M has the structure of an exact symplectic manifold. That is, it has a symplectic 2-form ω which is exact. In the case T^*M , $\omega = d\theta$, where θ is the Liouville one-form defined as follows. Let $p : T^*M \rightarrow M$ be the projection map. Let $x \in M$, and $t \in T_x^*M$. Then $\theta(x, t)$ is given by the composition,

$$\theta(x, t) : T_{x, t}(T^*M) \xrightarrow{dp} T_x M \xrightarrow{t} \mathbb{R}.$$

There has been a considerable amount of work comparing the symplectic topology of T^*M with the string topology of M . This relationship begins with a theorem of Viterbo [36], that the symplectic Floer homology is isomorphic to the homology of the free loop space,

$$SH_*(T^*M) \cong H_*(LM).$$

The symplectic Floer homology is computed via a Morse-type complex associated to the (possibly perturbed) ‘‘symplectic action functional’’, $\mathcal{A} : L(T^*M) \rightarrow \mathbb{R}$. The perturbation is via a choice of Hamiltonian, and so long as the Hamiltonian grows at least quadratically near infinity, the symplectic

Floer homology is described by the above isomorphism. The precise relationship between the Floer theory of the symplectic action functional \mathcal{A} and Morse theory on LM was studied in great detail by Abbondandolo and Schwarz in [1]. In particular they were able to show that a “pair of pants” (or “quantum”) product construction in $SH_*(T^*M)$ corresponds under this isomorphism to a Morse-theoretic analogue of the Chas-Sullivan product in $H_*(LM)$. In [14] this product was shown to agree with the Chas-Sullivan construction.

The objects of the Fukaya category $Fuk(T^*M)$ are exact, Lagrangian submanifolds $L \subset T^*M$. The morphisms are the “Lagrangian intersection Floer cochains”, $CF^*(L_0, L_1)$. These Floer cochain groups are also a Morse type cochain complex associated to a functional on the path space,

$$\mathcal{A}_{L_0, L_1} : \mathcal{P}_{L_0, L_1}(T^*M) \rightarrow \mathbb{R}.$$

If L_0 and L_1 intersect transversally, then the critical points are the intersection points (viewed as constant paths), and the coboundary homomorphisms are computed by counting holomorphic disks with prescribed boundary conditions. Of course if \mathcal{A}_{L_0, L_1} were actually a Morse function, satisfying the Palais-Smale convergence conditions, then these complexes would compute $H_*(\mathcal{P}_{L_0, L_1}(T^*M))$. One knows that this Morse condition is *not* satisfied, but there are examples, when this homological consequence is nonetheless satisfied. Namely, let $N \subset M$ be an oriented, closed submanifold. Let ν_N be the conormal bundle. That is, for $x \in N$, $\nu_N(x) \subset T_x^*M$ consists of those cotangent vectors which vanish on the subspace $T_x N \subset T_x M$. Notice that the conormal bundle is always an n -dimensional submanifold of the $2n$ -dimensional manifold T^*M . It is a standard fact that the conormal bundle $\nu_x N$ is a (noncompact) Lagrangian submanifold of T^*M . Notice that for any two closed, oriented submanifolds $N_0, N_1 \subset M$ the following path spaces in the cotangent bundle and in the base manifold M are homotopy equivalent:

$$\mathcal{P}_{\nu_{N_0}, \nu_{N_1}}(T^*M) \simeq \mathcal{P}_{N_0, N_1}(M).$$

The following was recently proven by Abbondandolo, Portaluri, and Schwarz [3]:

Theorem 14. *Given any closed, oriented submanifolds $N_0, N_1 \subset M$, then the intersection Floer cohomology of the $HF^*(\nu_{N_0}, \nu_{N_1})$ is isomorphic to the homology of the path space,*

$$HF^*(\nu_{N_0}, \nu_{N_1}) \cong H_*(\mathcal{P}_{N_0, N_1}(M)).$$

If one could realize these isomorphisms on the level of chain complexes, in such a way that the compositions correspond, then one would have a proof of the following conjecture:

Conjecture 15. *Let $Fuk_{conor}(T^*M)$ be the full subcategory of the Fukaya category generated by conormal bundles of closed, connected, submanifolds of M . Then there is a Quillen equivalence of categories with the string topology category,*

$$Fuk_{conor}(T^*M) \simeq \mathcal{S}_M.$$

Remarks.

1. In this conjecture one probably wants to study the “wrapped” Fukaya category as defined by Fukaya, Seidel, and Smith in [20].
2. When $N = \text{point}$, its conormal bundle is a cotangent fiber, $T_x M$. Abouzaid [4] has recently described the A_∞ relationship between the Floer cochains $CF^*(T_x M, T_x M)$ and the chains of the based loop space, $C_*(\Omega M)$.

There are other potential relationships between the Fukaya category and the string topology category as well. For example, Fukaya, Seidel, and Smith [20] as well as Nadler [29] building on work of Nadler and Zaslow [30] showed that when M is simply connected, the Fukaya category $Fuk_{cpt}(T^*M)$ generated by compact, exact Lagrangians, has a fully faithful embedding into the derived category of modules over the Floer cochains, $CF^*(M, M)$ where M is viewed as a Lagrangian submanifold of T^*M as the zero section. Furthermore, one knows that the Floer cohomology, $HF^*(M, M)$ is isomorphic to $H^*(M)$, and recently Abouzaid [5] proved that $CF^*(M, M) \simeq C^*(M)$ as A_∞ -differential graded algebras. So $Fuk_{cpt}(T^*M)$ can be viewed as a sub- A_∞ -category of the derived category of $C^*(M)$ -modules.

When M is simply connected, the string topology category \mathcal{S}_M can also be viewed as a subcategory of the category of $C^*(M)$ -modules. This is shown by using a generalized Poincare duality argument to prove the following:

Lemma 16. *Let M be a closed, simply connected manifold, and $N_0, N_1 \subset M$ connected, oriented, closed submanifolds. Then there is a chain equivalence,*

$$C_*(\mathcal{P}_{N_0, N_1}) \simeq \text{Rhom}_{C^*(M)}(C^*(N_1), C^*(N_0)).$$

Equivalently, there is a chain equivalence

$$\text{Rhom}_{C_*(\Omega M)}(C_*(F_{N_0}), C_*(F_{N_1})) \simeq \text{Rhom}_{C^*(M)}(C^*(N_1), C^*(N_0)).$$

The following is a strengthening of this lemma.

Theorem 17. *For M a closed, simply connected manifold, the string topology category \mathcal{S}_M is Quillen equivalent to the full A_∞ -subcategory of the derived category of $C^*(M)$ -submodules, generated by modules given by cochains $C^*(N)$ for $N \subset M$ a connected, oriented submanifold.*

From Nadler’s work we see that the relationship between the compact Fukaya category $Fuk_{cpt}(T^*M)$ should be equivalent to the “one-brane” string topology category, \mathcal{S}_M^M , which is the full subcategory of \mathcal{S}_M where the only D -brane is the entire manifold itself. This category, in turn, is equivalent to the derived category of perfect C^*M -modules.

The significance of this potential relationship is amplified when one considers recent work of Hopkins and Lurie [26] classifying “extended” topological conformal field theories. This can be viewed as a direct generalization of the work of Moore-Segal, and of Costello discussed above. In

their classification scheme, such a field theory is determined by an appropriately defined “dualizable object”. In the category of chain complexes, $C^*(M)$ is such an object. Thus the category of perfect $C^*(M)$ - modules should determine an extended field theory, which should correspond to string topology. On the other hand, by the above remarks, the Fukaya category $Fuk_{cat}(T^*M)$ should also determine a field theory, presumably the “Symplectic Field Theory” of Eliashberg, Givental, and Hofer [19]. One can therefore speculate that this line of reasoning may produce an equivalence of the symplectic field theory of T^*M , and of the string topology of M . There is evidence that such an equivalence may exist, for example the work of Cieliebak and Latshev [8]. Pursuing this relationship using the Hopkins-Lurie classification scheme could lead to a very satisfying understanding of the deep connections between these two important theories.

References

- [1] A. Abbondandolo and M. Schwarz, *On the Floer homology of cotangent bundles*, Comm. Pure Appl. Math. **59**, (2006) , 254–316.
- [2] A. Abbondandolo and M. Schwarz *Floer Homology of cotangent bundles and the loop product*, preprint MPI MIS no. 41, May 2008.
- [3] A. Abbondandolo, A. Portaluri, and M. Schwarz, *The homology of path spaces and Floer homology with conormal boundary conditions* Journal of Fixed Point Theory and Applications 4 (2008), 263-293. math.SG. arXiv:0810.1977
- [4] M. Abouzaid, *The wrapped Fukaya category and based loops*, preprint (2009).
- [5] M. Abouzaid, *A topological model for the Fukaya categories of plumbings*, preprint (2009), arXiv:0904.1474.
- [6] N. Baas, R.L. Cohen, and A. Ramirez, *The topology of the category of open and closed strings*, accepted for publication, Proc. of conf. in honor of S. Gitler, Contemporary Mathematics, (2005). preprint: math.AT/0411080
- [7] A. Blumberg, R.L. Cohen, and C. Teleman, *String topology, Morita theory, and Hochschild homology*, in preparation.
- [8] K. Cieliebak and J.Latshev, *The role of string topology in symplectic field theory*, preprint (2007), arXiv:mathSG0706.3284
- [9] M. Chas and D. Sullivan, *String Topology*. To appear in Ann. of Math. (preprint: math.GT/9911159).
- [10] R.L. Cohen, *The Floer homotopy type of the cotangent bundle*, accepted for publication in Pure and Applied Mathematics Quarterly, (2008), for the upcoming special issue in honor of M. Atiyah and I. Singer, preprint: math.AT/0702852

- [11] R.L. Cohen and V. Godin, *A polarized view of string topology*, Topology, Geometry, and Quantum Field Theory, London Math. Soc. Lecture Notes **vol. 308**, (2004), 127-154.
- [12] R.L. Cohen and J.D.S. Jones, *A homotopy theoretic realization of string topology*, Math. Annalen, **vol. 324**, 773-798 (2002).
- [13] R.L. Cohen, J. Klein, and D. Sullivan, *The homotopy invariance of the string topology loop product and string bracket* preprint: math.GT/0509667
- [14] R.L. Cohen and M. Schwarz, *A Morse theoretic description of string topology*, to appear in Proc. of Conference on Symplectic Field Theory in honor of the 60th birthday of Y. Eliashberg, (2009). preprint: arXiv:0809.0868
- [15] R.L. Cohen, and J. Klein *The Pontrjagin-Thom construction and homotopy pullbacks*, Stanford University preprint, (2005).
- [16] R.L. Cohen and A. Voronov, *Notes on String Topology*, to appear in CRM Lecture Notes from summer school on string topology and Hochschild homology, Almeria Spain (2005), preprint: math.GT/0503625
- [17] K. Costello, *Topological conformal field theories and Calabi-Yau categories*. Adv. Math. 210 (2007), no. 1, 165–214.
- [18] W. Dwyer, J. Greenlees, and S. Iyengar, *Duality in algebra and topology*, Adv. Math. 200(2006), no. 2, 357-402. preprint: math.AT/0510247
- [19] Y. Eliashberg, A. Givental, and H. Hofer, *Introduction to symplectic field theory*, Geom. Funct. Anal., special vol. part II, (2000), 560-673
- [20] K. Fukaya, P. Seidel, and I. Smith *The symplectic geometry of cotangent bundles from a categorical viewpoint* preprint (2007), arXiv:0705.3450.
- [21] E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Comm. Math. Phys., **159 (2)** (1994), 265-285.
- [22] V. Godin, *Higher string topology operations*, preprint arXiv:0711.4859
- [23] E. Harrelson, *On the homology of open-closed string field theory*, (2004) preprint math.AT/0412249
- [24] J. Klein, *The dualizing spectrum of a topological group* *Math. Annalen* **319**, 421–456 (2001)
- [25] J. Klein, *Fiber Products, Poincar duality and A_∞ -ring spectra*, Proc. Amer. Math. Soc. 134 (2006), no. 6, 1825–1833
- [26] J. Lurie, *On the classification of topological field theories*, preprint, (2009).

- [27] E. Malm, *Ph.D thesis, Stanford University*, in preparation
- [28] G.W. Moore and G. Segal, *D-branes and K-theory in 2D topological field theory*, preprint: hep-th/0609042
- [29] D. Nadler, *Microlocal branes are constructible sheaves* Preprint, math.SG/0612399.
- [30] D. Nadler and E. Zaslow, *Constructible sheaves and the Fukaya category*, Preprint, math.SG/0406379
- [31] A. Ramirez, *Open-Closed String Topology via Fat Graphs*, (2006) preprint: math.AT/0606512
- [32] G. Segal, *Topological structures in string theory*, Phil. Trans. Royal Soc. Lond. A **359**, (2001), 1389-1398.
- [33] G. Segal, *The definition of conformal field theory*, in “Topology, Geometry, and Quantum Field Theory”, London Math. Soc. Lecture Note Series **308** (2004), 421-577,
- [34] A. Sen and B. Zwiebach, *Background independent algebraic structures in closed string field theory*, Comm. Math. Phys., **177**, (1996), 305-326.
- [35] D. Sullivan, *Open and closed string field theory interpreted in classical algebraic topology*, Topology, geometry and quantum field theory, London Math. Soc. Lecture Notes, vol. 308, Cambridge Univ. Press, Cambridge, 2004, math.QA/0302332, pp. 344–357.
- [36] C. Viterbo, *Functors and computations in Floer homology with applications, Part II*, Preprint, 1996.