Curved algebras and Modules

The Dirac family assignment $V \mapsto (V \otimes S^{\pm}; \not{D}_V + \psi(\xi))$ from G reps to (2-term) complexes of G-bundles on $\mathfrak{g}$, or from LG reps to twisted (cx. of) G-bdles on G, is not an equivalence of categories but can be refined to one in the world of curved algebras.

Definition. A curved dg algebra (A, d, W) is:

- an associative graded algebra A
- a degree 1 derivation $d : A \to A$
- an element $W \in A^{(2)}$ with $d^2 = [W, .]$ and $dW = 0$.

Examples

- If $d^2 = 0$, W can be any central element.
- Often, the grading is collapsed mod 2.
- X manifold, $A = \Omega^*(X)$, $d =$ usual, $\omega \in \Omega^2(X)_{cl}$
- X manifold, $E \to X$ vector bundle with connection, $A = \Omega^*(X; \mathrm{End}\, E)$, $d = \nabla_E$, $W = F_E$ curvature.

Definition* A curved dg module (M, ∇) over A is a graded A-module with d-compatible derivation (connection) ∇ satisfying $\nabla^2 = W$.

Remark One places finiteness conditions on M to make this useful (Positelski; Preygel) and can then construct a diff graded category of curved modules with associated derived category.

Examples

- Modules over $(\Omega^*(X, \mathrm{End}(E)), \nabla_E, F_E) \Longleftrightarrow$ (Flat) modules over $(\Omega^*(X), d, 0)$ by $\otimes E$.
- If $\omega \in \Omega^2(X)_{cl}$ has an integral lift, get $\Longleftrightarrow$

of ω-curved and flat $(\Omega^*(X), d)$-modules.

- $\mathrm{Spec}(A)$ smooth, char. 0: Orlov's theorem:

$$D((A,W)\text{-mod}) \cong D_{sing}(W^{-1}(0))$$
$$:= D(W^{-1}(0))/\mathrm{Perf}(W^{-1}(0)).$$

[Customary definition takes in all $W^{-1}(c)$ for critical values c of W].

Equivalence is established by: (P, Q A-projective)

$$\Big(P \underset{q}{\overset{p}{\rightleftarrows}} Q\Big) \longrightarrow \big(P \xrightarrow{p} Q\big)\big(\sim \mathrm{Coker}(p)\big)$$
$$\quad (A,W)\text{-mod} \qquad\qquad D(W^{-1}(0)), \text{ project to } D_{sing}$$

- $A = \mathbb{C}[x_1, \dots x_n]$, $W = x_1^2 + \dots + x_n^2 = q(x)$;

$(A,W)\text{-mod} \Longleftrightarrow \mathrm{Cliff}(n;q)\text{-mod}$

Morita equivalence defined by Atiyah-Bott-Shapiro Thom class in K-theory:

$$\mathrm{Cliff}^{ev}(n) \underset{\psi(x)\cdot}{\overset{\psi(x)\cdot}{\rightleftarrows}} \mathrm{Cliff}^{odd}(n) \quad \text{with right } \mathrm{Cliff}(n) \text{ action.}$$

Remark. The Thom class uses only one of the arrows.

Remark. Similar story for Morse-Bott functions.

Caution: Homological algebra with curved algebras / modules is very tricky.

Example: Quasi-isomorphisms of dga's do NOT induce derived equivalences of curved module cats. (Curved module categories are model-dependent).

Let $X = \mathbb{C}\mathbb{P}^\infty$; $(\Omega^*(X), d) \sim H^*(X) \cong \mathbb{C}[[\omega]]$.

But $(\Omega^*(X), d, \omega)$-mod is $\Leftrightarrow$ $(\Omega^*(X), d, 0)$-mod by $\otimes$ with $\mathcal{O}(-1)$; whereas $(\mathbb{C}[[\omega]], \omega)$-mod is the zero category. (Orlov).

(Works better when W is nilpotent.)

Key example (Curved Cartan complex)

G = cpt Lie group acting on manifold X

ξ_a basis of $\mathfrak{g}$, $\mathcal{L}_a$ = Lie action, ι_a = contraction

Consider the crossed product algebra

$$G \ltimes (\Omega^*(X) \otimes \mathrm{Sym}\, \mathfrak{g}^*)$$

$$d = d_X + \xi^a \otimes \iota_a \qquad \text{Cartan differential}$$

$$W = \xi_a(\delta_1) \otimes \xi^a \qquad \text{curvature!}$$

The meaning of the C.C. complex is best described in terms of group actions on categories: Its category of curved modules is $\Leftrightarrow$ G/G fixed category of $(\Omega^*(X), d)$-modules

Example: $G =$ torus T, $X =$ point

$$T \ltimes \mathrm{Sym}\,\mathfrak{t}^* = \bigoplus_\lambda \mathrm{Sym}\,\mathfrak{t}^* = \bigoplus_\lambda \mathrm{Funct}(\mathfrak{t}_\lambda)$$

(Copies of $\mathfrak{t}$ indexed by the characters of T)

W on $\mathfrak{t}_\lambda = \lambda$ (linear)

So, curved cartan mods = $\mathrm{Sym}\,\mathfrak{t}^*$-mod = $H^*(BT)$-mod

Example: Casimir twists

We can add arbitrary G-invariant functions on $\mathfrak{g}$ (= classes in $H^*(BG)$) as additional curvings in the Cartan complex.

Adding a quadratic function to $G = T$, $X =$ point creates ONE nondegenerate critical point on each $\mathfrak{t}_\lambda$

Theorem. The category of curved modules for the Casimir-curved Cartan complex of G ($X =$ point) $\overset{\sim}{\Longleftrightarrow}$ $\mathrm{Rep}(G)$, via the Dirac family construction

The category of curved modules for the Casimir-curved Cartan complex of $\widehat{LG}$ ($X =$ point) at nonzero levels $\overset{\sim}{\Longleftrightarrow}$ PE $\mathrm{Rep}(LG)$, again via the Dirac fam. construction.

Remarks. Additional curvings are possible and meaningful, The curved Cartan complexes for G and LG are tied to 2-dimensional TQFT's controlling the topology of flat G-bundles on surfaces.

More precisely, their categories of modules are the outputs of a point in this 2dim TQFT.

The invariants for surfaces are closely related to

i integration over the moduli of flat connections for G

ii index theory (K-integration) for LG.

(i) - conjectured by Witten (1990) Jeffrey-Kirwan sums

(ii) - conjectured by -, proved w/ Woodward
independently discovered by Nekrasov-Shatashvili
(as 1-2 dim theory only)

Ingredients An integration class for Bun_G arises from a characteristic class (in $H^{ev}(BG)$) of the universal bundle over $\Sigma \times \mathrm{Bun}_G$, by integrating over Σ and exponentiating. ($H^4 \leftrightarrow$ Casimir twists).

A K-integration (index) class for Bun_G (stack) arises from one in $K_G(pt) = \mathrm{Rep}(G)$ by taking the index along Σ and exponentiating. Line bundles come from H^4

Others - $\mathrm{Rep}(G) \otimes \mathbb{C} = \mathbb{C}[G/G]$, use as loop group twists.

Group actions on Categories, and interpretation of the Curved Cartan Complex

Definition An action of a group G on a category $\mathcal{C}$ consists of:

- a functor $\Phi_g : \mathcal{C} \to \mathcal{C}$ for each $g \in G$
- an isomorphism $\alpha_{g,h} : \phi_g \circ \phi_h \xrightarrow{\sim} \phi_{gh}$ for each pair $g, h \in G$

subject to an associativity constraint

$$\begin{tikzcd}
\phi_g \circ \phi_h \circ \phi_k \arrow[r, "\phi_g(\alpha_{h,k})"] \arrow[d, "\alpha_{g,h} \circ \phi_k"'] \arrow[dr, phantom, "\circlearrowright"] & \phi_g \circ \phi_{hk} \arrow[d, "\alpha_{g,hk}"] \\
\phi_{gh} \circ \phi_k \arrow[r, "\alpha_{gh,k}"'] & \phi_{ghk}
\end{tikzcd}$$

Examples.
- G acts on the cat. of Vector bundles on G
- Likewise for flat bundles or $(\Omega^*(G), d)$-modules
- Likewise for $Vect(X)$, if G acts on X
- If $G \to Aut(A)$, then G acts on A-mod.
- T acts on $Coh(T^\vee)$ via the Poincaré bundle on $T \times T^\vee$. This action is in fact trivializable when lifted to $\mathfrak{t}$. ($\Leftarrow$ the Poincaré bundle can be given a flat structure along T.)

This is the regular locally trivial representation of T in linear categories.

Topological, differentiable, analytic, algebraic actions

There is a uniform method due to Grothendieck teaching us to encode structure on X via the functor Hom(. ; X) on the category of top, smooth or analytic spaces. (This functor is a sheaf

So a structure on a category $\mathcal{C}$ will be an enrichment of $\mathcal{C}$ to a sheaf in categories on the respective site.

Remark: Better to use quasicategories and the model structures on them. (Joyal; Tierney; Boardman,

A category with structured G action will be a sheaf of categories on the site of spaces with struct. G-action

More generally, we can talk about sheaves or bundles of categories over spaces, with connections, flat connections, coherent sheaves of .., G-equivariant ones, etc.

Categories with G-action $\leftrightarrow$ categories over BG

with G/G action $\longleftrightarrow$ w/ flat connection

with locally trivial G action $\longleftrightarrow$ locally constant

Fixed point category $\longleftrightarrow$ Global sections over BG

Derived —"—"— $\longleftrightarrow$ Cohomology over BG

Fixed point category (discrete group case)

$$\mathcal{C}^G := \left\{ (x, \varphi_g) \;\middle|\; x \in \mathrm{Ob}\,\mathcal{C};\ \varphi_g : x \xrightarrow{\sim} \Phi_g x \right\}$$

such that composition ...

Example: · For the trivial G-action on Vect,

$\mathrm{Vect}^G = \mathrm{Rep}(G)$.

For the G-action on Vect described by $\tau \in H^2_G(\mathcal{O}^\times)$,

$\mathrm{Vect}^G = \tau$-projective reps of G.

Remark: τ appears in the action as a 2-cocycle

$\alpha_{g,h} : G \times G \longrightarrow \mathbb{C}^\times$.

· Same stories for Lie & algebraic groups

· For $G/\hat{G}$ (infinitesimally trivialised action on Vect,

$\mathrm{Vect}^{G/\hat{G}}$ = modules over the de Rham complex of G with the convolution structure

(= group algebra of $G/\hat{G}$)

$\overset{\text{Koszul}}{\longleftrightarrow}$ Modules over curved Cartan cx $G \ltimes \mathrm{Sym}\,\mathfrak{g}^*$ (deg 2)

$\longleftrightarrow$ Modules over $(C_* G, \partial)$ for any model of chains

$\overset{\text{if } G \text{ connected}}{\longleftrightarrow}$ modules over $C^*(BG) \sim H^*(BG)$.

· For the locally trivialised action of G on Vect:

Same answers

["$\mathcal{D}$-modules on the stack BG are integrable"]

Caution: Mind ... Koszul duality

Actions on Vect

Theorem. Locally-trivialized actions of T on Vect $\longleftrightarrow$ points in the Langlands dual torus $T^\vee$. $\langle H^2(BT;\mathbb{C}^\times)\rangle$

This hints at the moral interpretation of the Langlands dual group $G^\vee$: Conjugacy classes in $G^\vee$ would like to be "one-dimensional locally trivialized categorical reps of G." Not quite true - the latter must form a group - but there is a picture close to this.

Proof. T acts on Vect, trivialized lift to $\mathfrak{t}$
$\Longleftrightarrow$ two trivializations of the action to $\exp^{-1}(1) = \pi_1 T$
$\Longleftrightarrow$ action of $\pi_1 T$ by automorphisms of $\mathrm{Id}_{\mathrm{Vect}}$
$\Longleftrightarrow$ 1-dim complex rep of $\pi_1 T$
$\Longleftrightarrow$ point in $T^\vee$.

Remark. This proves that categories with locally-trivial T-action are the same as categories fibered over $T^\vee$, or module categories over $\mathrm{Coh}(T^\vee)$ (if closed under colimits)

Yet again: have an E_2 ring homomorphism
$$\mathbb{C}[\pi_1 T]^\vee \cong \mathbb{C}[T^\vee] \longrightarrow HH^0(\mathcal{C}).$$

If G is simply connected nonabelian, argument shows a locally trivial action on Vect is trivial.

But there exist interesting actions on (diff) graded vector spaces, or "$\mathbb{Z}/2$ graded" actions on Vect.

$\to$ classified by $H^2(BG; \mathbb{C}[[q]]^\times)$ $(\deg q = -2)$

Theorem. Locally trivial actions of G on $\mathcal{C}$

$\longleftrightarrow$ module structures of $\mathcal{C}$ over $(\mathrm{DerLoc}(G), *)$

(derived cat. of local systems on G w. convolution tensor structure)

$\longleftrightarrow$ E_2-alg. homomorphisms $H_*(\Omega G; \mathbb{C}) \to HH^*(\mathcal{C})$

if G is connected.

Remark: $\operatorname{Spec} H_*(\Omega T; \mathbb{C}) = T^\vee$.

· A theorem of Bezrukavnikov, Finkelberg, Mirković identifies $\operatorname{Spec} H_*(\Omega G; \mathbb{C})$ with an exceptional fibre in a blow-up of $T^*(G^\vee/G^\vee)$. $((T^*G^\vee)^{reg}/\!/G^\vee)$.

There is a semiclassical picture for categories over this in terms of Lagrangians in the cotangent bundle, fixed-point categories in terms of intersections, etc.

[under construction]

"Theorem." Modules over the Casimir-curved Cartan model for G are the "non-perturbative" fixed point category for the ($\mathbb{Z}/2$-graded) locally trivial action of G on Vect defined by the "Casimir class" in $H^4(BG)$. (Additional curvings in $H^{ev}(BG;\mathbb{C})$ give perturbatively computable deformations thereof).

The analogue holds for LG and its twistings.

Remark. This is largely a definition. The theorem consists in the claim that the thus defined categories generate "susy Yang-Mills theory" in 2 dims, for G and LG respectively, which control integration/index formulae over Bun_G.

A "perturbative" computation of the fixed-pt categories would start with zero curvature, giving $H^*(BG)$-mod, and add a Casimir curvature in H^4, giving Vect for a torus (Morse critical point) and 0 for nonab. Neither of these can reproduce the integration formulas for Bun_G correctly!