DE RHAM'S THEOREM FOR ∞-STACKS

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Abstract. Someday there may be one.

1. Introduction

Simplicial constructions seem to have debuted in algebraic geometry with Deligne's mixed Hodge theory for singular varieties [D2]: a mixed Hodge structure arises naturally when resolving a singular variety (in a suitable sense) by a smooth simplicial one, with the same total cohomology. Since, however, simplicial substitutes cannot be chosen functorially, a key step in the construction is to check independence of the choices. In other words, the desired cohomology functors, defined a priori on the category of simplicial schemes, must factor through a *homotopy category*, obtained from the former by inverting a certain class of morphisms. In the homotopy category, the original variety becomes isomorphic to its simplicial resolution. Convention calls morphisms in a class to be inverted *weak equivalences*.

Based on work of Verdier [SGA4], Deligne chose the class of proper hypercoverings (see Sect.2) as weak equivalences. Verdier showed that cohomology with (locally) constant coefficients factors through the homotopy category [D2], Sect.6. This suffices for the purpose at hand (see Prop. 2.7); but, for a technical reason which homotopy theorists will appreciate[1], in full generality hypercoverings are not quite the "right" notion. Rather, the latter was proposed by Illusie [I] and (independently) by Brown [B] (cf. Def. 2.2). Its moral core became apparent when Joyal proved that the Illusie weak equivalences (IWEs) are part of a *Quillen homotopy structure*, on simplicial objects in any Grothendieck topos [J1], [Jo]. (A special case had been established by Brown and Gersten in [BG]). Inverting IWEs results in the *homotopy category of stacks* (better, ∞-stacks) over the topos in question[2]. The homotopical algebra of [Q] allows a calculus with stacks that strongly mirrors ordinary homotopy theory, and is also analogous to working in the derived category of an Abelian category. This material has recently recieved significant exposure through the work of Morel and Voevodsky [MV], which shows convincingly how a category of mixed motives can be constructed using these homotopical techniques.

Our reasons for looking at this theory (which date from before the appearance of [MV], cf [Si3], [Si2], [T1]) are related to two directions of generalization of the notion of de Rham cohomology of an algebraic variety. These two directions can be summed up by saying that we would like to put stacks in the coefficients, and in the domain object of the de Rham cohomology functor. The former is treated in [Si2] and more recently [Si4] and we won't discuss it at any great length here. The latter was suggested in 1996 by the second author of the present paper and is the main subject of our discussion.

The idea is to look at the de Rham cohomology of a stack. This comes with the usual associated baggage of Hodge structures and the Hodge-to-de Rham spectral sequence. They are all features of a stack, not of a particular simplicial representative. Furthermore, K-theory and various *motivic realizations* [Hu] also descend to appropriate categories of stacks (indeed, all functors which turn hypercoverings into isomorphisms do so), but those are better left to experts.

Date: 1997?

1991 *Mathematics Subject Classification.* Primary ?; Secondary ?

Key words and phrases. Stacks, Grothendieck sites, Simplicial objects.

[1]Closely related to the *Kan condition* in the homotopy theory of simplicial sets

[2]The choice of a Grothendieck topology on the category in question is necessary, and affects the outcome

The first examples which one comes across concern the case of the de Rham cohomology of a 1-stack such as the moduli stack of G-bundles over a curve. We will look explicitly at these examples which first arose in [T1]. Simple enough to give nice answers, these examples do benefit from general nonsense: as they are objects of infinite type, the bare-handed treatment is somewhat tedious.

The introduction of simplicial methods as the easiest way of dealing with 1-stacks naturally leads to the idea of applying the same methods to all "∞-stacks" or simplicial presheaves (or, as we call them in the present paper, just stacks). We don't for the moment have any concrete applications to propose here but in the expectation that such examples will arise sometime, we develop the theory in this generality.

The material in Sect.2 greatly overlaps with [MV]; we deem it necessary to apologize for that, but feel that a concise exposition of the topic can be beneficial. (Most importantly, it had already been written, so there was little sense in deleting it). At any rate, we claim little originality here, this being mostly a restatement of work of Illusie, Brown, Joyal and Jardine, as the references make clear.

2. Illusie's weak equivalences and the model structure

2.1. **Basic definitions.** In this section, $\mathfrak{C}$ is a site, a small category with a Grothendieck topology; $\mathbb{S}Sh\,(\mathfrak{C})$ will denote the category of sheaves of simplicial sets over $\mathfrak{C}$. For background on simplicial objects, see [BK], [K] or the first two chapters of [M]; for the basics of Grothendieck topologies, see [G]. To simplify matters, we shall assume that all representable functors on $\mathfrak{C}$ are sheaves (cf. [G]), and that $\mathfrak{C}$ *has enough stalks*. Here, a *stalk* is an exact functor p^* from $Sh\,(\mathfrak{C})$ to the category of sets which admits a right adjoint p_*. By definition, $\mathfrak{C}$ has enough stalks when monomorphisms and epimorphisms of sheaves can be detected on stalks. When $\mathfrak{C}$ is the category of open sets in a reasonable[3] topological space, we recover the usual notion of stalks, corresponding to points in the space; and there are enough of them. The stalk of a simplicial sheaf is a simplicial set, and a morphism of simplicial sheaves induces morphisms on all stalks.

2.2. Definition. (cf. [I], [B], [BG], [J1]) Let $\varphi : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of simplicial sheaves.

(i) φ is an *Illusie weak equivalence (IWE)* if it induces a weak homotopy equivalence on all stalks.

(ii) φ is a *cofibration* iff it is a monomorphism.

(iii) φ is a *(global) fibration* if it satisfies the "right lifting property" ([Q], Sect. 1.1, see also (2.3) below) with respect to all weakly equivalent cofibrations.

(iv) φ is a *Kan (or local) fibration* if it induces Kan fibrations on all stalks.

(v) φ is a *hypercovering* if it is a Kan fibration with stalkwise contractible fibers.

2.3. Remark. Recall [K] that a map $f : X \to Y$ of simplicial sets is a (Kan) fibration if, for any weakly equivalent inclusion $S \subseteq T$ of finite simplicial sets, the naturally induced map $\mathrm{Hom}\,(T; X) \to \mathrm{Hom}\,(S; Y)$ is surjective. This, incidentally, is the "right lifting property" with respect to the inclusion $S \subseteq T$. In this spirit, condition (iv) can be restated as follows: regarding S and T as constant simplicial sheaves over $\mathfrak{C}$, the induced morphism on sheaves $\mathcal{H}om\,(T; \mathcal{X}) \to \mathcal{H}om\,(S; \mathcal{Y})$ must be epic (i.e. a covering, or surjective on stalks). $\mathcal{H}om$ is the sheafified Hom on sheaves of simplicial sets. For (v), we must drop the requirement that the inclusion should be a weak equivalence, or else, equivalently, add the condition that φ be an IWE. There is also a stalk-free way to define global fibrations and weak equivalences (cf. [J1]), and the sufficient stalks restriction on $\mathfrak{C}$ can be removed.

2.4. Theorem. *(Joyal; cf.* [BG], [J1], [Jo]*). The category* $\mathbb{S}Sh\,(\mathfrak{C})$*, with cofibrations, IWEs and global fibrations as defined, satisfies Quillen's axioms for a closed simplicial model category.* ∎

Let $\mathrm{Ho}\,(\mathfrak{C})$ be the homotopy category arising by inverting the weak equivalences in $\mathbb{S}Sh\,(\mathfrak{C})$. This is our category of stacks (or ∞–stacks) over $\mathfrak{C}$. $\mathrm{Hom}\,(\mathcal{X}; \mathcal{Y})$, in $\mathrm{Ho}\,(\mathfrak{C})$, is denoted by $[\mathcal{X}; \mathcal{Y}]$,

[3]See the definition of a *sober* space in [SGA4]. Hausdorff spaces qualify, as do schemes in the Zariski topology.

or Ext $(\mathcal{X};\mathcal{Y})$ (by analogy with the derived category notation). Of course, Ho := Ho (point) is the ordinary homotopy category. We can arrange for Ho $(\mathfrak{C})$ to have the same objects as $\mathbb{S}Sh(\mathfrak{C})$, so a stack "is", in a sense, a simplicial sheaf. But this point of view has its limitations: just as a topological homotopy type can be indicated in numerous ways, many natural constructions would produce the same Ho $(\mathfrak{C})$. For instance, sheaves and presheaves of topological spaces are used in [Si2]. One could also consider arbitrary small diagrams of objects in $Sh(\mathfrak{C})$, instead of just simplicial diagrams; the associated stack in Ho $(\mathfrak{C})$ is the homotopy direct limit of the diagram, and can be represented simplicially by a standard, functorial construction (see [BK] , Ch.XII). Finally, although less obviously, one may use diagrams of objects from $\mathfrak{C}$ itself; this is part of the "hypercovering theorem" (2.7) below.

2.5. **Geometric constructions.** Stalks commute with finite inverse limits and arbitrary direct limits. Constructions in simplicial homotopy theory which involve only such limits (called *geometric constructions*) can be performed stalkwise, and share the properties of the set-theoretic counterparts. One example is Kan's functor $\mathrm{Ex}^{\infty} := \lim_{n\to\infty}\mathrm{Ex}^n$, which produces a Kan object; Ex is the right adjoint of barycentric subdivision in the category of simplicial sets, and only involves finite limits [K]. On simplicial sheaves, Ex and Ex^{∞} may be defined stalkwise, or, equivalently, using the sheaf direct and inverse limits. We thus get, for any $\mathcal{X}$, a functorial, Illusie weakly equivalent "Kanification" $\mathcal{X} \longrightarrow \mathrm{Ex}^{\infty}\mathcal{X}$, with $\mathrm{Ex}^{\infty}\mathcal{X}$ locally fibrant. More generally, any morphism $\varphi : \mathcal{X} \longrightarrow \mathcal{Y}$ factors through a locally fibrant "relative Kanification morphism" $\mathrm{Ex}^{\infty}\varphi : \mathrm{Ex}^{\infty}_{\mathcal{Y}}\mathcal{X} \longrightarrow \mathcal{Y}$ of $\mathcal{X}$ over $\mathcal{Y}$. If φ was an IWE, $\mathrm{Ex}^{\infty}\varphi$ is a hypercovering of$\mathcal{Y}$ ([F], Ch.I).

Another geometric construction is the *Postnikov tower* $\mathcal{X}^{\leq n}$ of a stack $\mathcal{X}$. There is a natural tower of morphisms $\mathcal{X} \longrightarrow \mathcal{X}^{\leq n}$. However, without some assumption on the ground site, this need not give a weak equivalence of $\mathcal{X}$ with the homotopy inverse limit of $\mathcal{X}^{\leq n}$ (rather, the latter is a sort of completion thereof), the problem being that infinite inverse limits need not commute with stalk formation. (This observation is implicit in [J1]; it appears explicitly in [MV], §2.1, with a nice example). If, however, the functor of sections has finite cohomological dimension, locally on $\mathfrak{C}$ (meaning that, for all objects U of a generating subcategory of $\mathfrak{C}$, the functor $\mathbb{R}\Gamma(U;-)$ has finite cohomological dimension), then the sheafified homotopy groups of $\mathrm{holim}_{n\leftarrow}\mathcal{X}^{\leq n}$ do agree with those of $\mathcal{X}$. This follows from the *Bousfield-Kan spectral sequence* for the homotopy groups of $\mathrm{Ext}(U;\mathcal{X})$, which converges completely, under this finiteness assumption. (That spectral sequence is a special case of the Leray sequence in [J2]). The assumption is verified on the Zariski and analytic sites; but the étale site requires some care.

2.6. **Kan objects and hypercoverings.** In the examples of interest, where $\mathfrak{C}$ is a category of schemes or of analytic spaces, global fibrations and globally fibrant objects are not too appealing (they are analogous to flabby sheaves); but Kan fibrations are quite natural, as are Kan objects (simplicial sheaves Kan fibered over a point). For instance, with a group G acting on a scheme X, the simplicial homotopy quotient ("bar construction") is a Kan object. In fact, for any simplicial group sheaf $\mathcal{G}$, the bar construction of $B\mathcal{G}$ (see e.g. [M], §21) is locally fibrant. Similarly, any Artin stack (i.e. algebraic groupoid $X_1 \rightrightarrows X_0$, with smooth source and target morphisms) leads to the locally fibrant "classifying" simplicial sheaf $\mathcal{X}$, represented by its nerve, in which $\mathcal{X}_n = X_1 \times_{X_0} X_1 \times_{\dots} \times_{X_0} X_1$ (n factors; a description of the simplicial maps can be found in [S]).

Let $\mathbb{S}_K\mathfrak{C} \subset \mathbb{S}Sh(\mathfrak{C})$ be the full subcategory of simplicial Kan objects that are, dimensionwise, represented by direct sums of objects in $\mathfrak{C}$. For a Kan object $\mathcal{X} \in \mathbb{S}Sh(\mathfrak{C})$, let $HC(\mathcal{X})$ be the following category. Objects are the hypercoverings $\alpha : \mathcal{U} \to \mathcal{X}$, with $\mathcal{U}$ in $\mathbb{S}_K\mathfrak{C}$. The set of morphisms from $(\alpha : \mathcal{U} \to \mathcal{X})$ to $(\beta : \mathcal{V} \to \mathcal{X})$ is the quotient of the set $\{\varphi : \mathcal{U} \to \mathcal{V} \mid \beta\circ\varphi = \alpha, \text{up to simplicial homotopy}\}$, modulo the relation generated by simplicial homotopies. (Note that this equivalence relation is compatible with composition). Let $\mathrm{Hom}_\bullet(\mathcal{X};\mathcal{Y})$ be the simplicial Hom, defined, as usual, by $\mathrm{Hom}_k(\mathcal{X};\mathcal{Y}) = \mathrm{Hom}\left(\mathcal{X}\times\Delta^k;\mathcal{Y}\right)$.

2.7. Theorem. *("Verdier Hypercovering theorem",* [SGA4], [B]*).*

(i) $HC(\mathcal{X})$ *is left filtering, and* $[\mathcal{X};\mathcal{Y}] = \lim_{\mathcal{U}\in HC(\mathcal{X})\to} \pi_0\mathrm{Hom}_\bullet(\mathcal{U};\mathcal{Y})$.

(ii) The natural functor from $\mathbb{S}_K\mathfrak{C}$ *to* $\mathrm{Ho}(\mathfrak{C})$ *becomes an equivalence of categories, once the hypercoverings in* $\mathbb{S}_K\mathfrak{C}$ *are inverted.* ∎

2.8. Remark. (i) This formulation of the theorem is essentially due to Brown ([B], Thm.1). Verdier gave a method to produce $\mathfrak{C}$-hypercoverings (this is also summarized in [D2], §6.2), and proved (ii) with $\mathcal{Y} = K(\mathcal{A}; n)$, for Abelian $\mathcal{A}$, as the correct generalization of Čech cohomology. When $\mathcal{Y} = K(\mathcal{A}; n)$, the hypercohomology $\mathbb{H}^n(\mathcal{X}; \mathcal{A})$ replaces $[\mathcal{X}; \mathcal{Y}]$; see [B], Prop.5.

(ii) Recall that a *generating subcategory* $\mathfrak{G}$ for $Sh(\mathfrak{C})$ is a small, full subcategory whose objects can be used to cover any object in $\mathfrak{C}$. Then, $\mathfrak{G}$ inherits a Grothendieck topology from $\mathfrak{C}$, and restriction from $\mathfrak{C}$ to $\mathfrak{G}$ gives an equivalence on the categories of sheaves. In that case, $\mathrm{Ho}(\mathfrak{G})$ and $\mathrm{Ho}(\mathfrak{C})$ are also equivalent. Any generating subcategory could be used in the theorem, instead of $\mathfrak{C}$.

2.9. **Morphisms and derived morphisms of topoi.**

2.10. Definition. Given two sites $\mathfrak{B}$ and $\mathfrak{C}$, a *morphism of topoi* $\Phi : Sh(\mathfrak{B}) \longrightarrow Sh(\mathfrak{C})$ is a pair of functors $\Phi^* : Sh(\mathfrak{C}) \longrightarrow Sh(\mathfrak{B})$, $\Phi_* : Sh(\mathfrak{B}) \longrightarrow Sh(\mathfrak{C})$, with Φ^* left exact and left adjoint to Φ_*.

Typically, such functors arise in one of two manners [G]:

(a) From a functor $\varphi^{-1} : \mathfrak{C} \longrightarrow \mathfrak{B}$ which preserves coverings and fibered products (assuming that finite fibered products exist in $\mathfrak{B}$ and $\mathfrak{C}$).

(b) From a functor $\varphi : \mathfrak{B} \longrightarrow \mathfrak{C}$ with the property that, for any $b \in \mathfrak{B}$ and any covering $\alpha : c \to \varphi(b)$ in $\mathfrak{C}$, there exist coverings $\beta : \tilde{b} \to b$ in $\mathfrak{B}$ for which $\varphi(\beta)$ factors through α. (One simply says that φ *pulls back coverings to coverings*).

2.11. Remark. (i) The criteria are explained by the fact that a functor $f : \mathfrak{B} \longrightarrow \mathfrak{C}$ always induces an adjoint triple (F_+, F^{-1}, F_*) on *presheaves.* Part (a) ensures that F^{-1} preserves the sheaves and that the sheafification of F_+ is exact; the latter becomes Φ^*, while $\Phi_* = F^{-1}$. Part (b) guarantees that F_* preserves the sheaves, in which case, it is our Φ_*; while Φ^* is the sheafification of F^{-1}.

(ii) In (a), Φ^* extends φ^{-1} to sheaves, and the direct image sheaf is $\Phi_*\mathcal{F}(c) = \mathcal{F}\left(\varphi^{-1}c\right)$; but in (b), Φ_* is not an extension of φ. Rather, φ sometimes extends to a *left* adjoint functor $\Phi_!$ to Φ^* (see the third example below).

(iii) If we only assume in (a) that φ^{-1} preserves fibered products of the form $U \times_V U$, for arbitrary V of $\mathfrak{C}$ but only a cofinal collection of coverings $U \to V$, we still get an adjoint pair (Φ^*, Φ_*) on sheaves; but Φ^* need not be left exact anymore.

2.12. Example. (i) $\mathfrak{B}$ and $\mathfrak{C}$ are the categories of disjoint unions of open sets in topological spaces B and C, $\varphi : B \longrightarrow C$ a continuous map, φ^{-1} is the inverse image. Construction (a) recovers the usual operations on sheaves.

(ii) As before, but with $\varphi : B \longrightarrow C$ an *open* map: this time, (b) recovers the usual φ^* and φ_*.

(iii) As before, but with $\varphi : B \longrightarrow C$ an open *embedding.* In this case, both (a) and (b) are satisfied, and we get the triple $(\Phi_!, \Phi^*, \Phi_*)$, the leftmost being extension by zero.

(iv) $\mathfrak{B}$ = analytic spaces in the classical topology, $\mathfrak{C}$ = complex schemes of finite type in the Zariski (or étale) topologies; φ^{-1} is the "underlying analytic space" functor. $\Phi^*(X)$ will be denoted X^{an}.

(v) $\mathfrak{B}$ = schemes in the étale topology, $\mathfrak{C}$ = schemes in the Zariski topology, φ^{-1} = identity.

(vi) $\mathfrak{B}$ = schemes in the topology generated by étale coverings and by proper, surjective maps, $\mathfrak{C}$ = schemes in the étale topology, φ^{-1} = identity.

(vii) As in (vi), but using analytic spaces (where the étale topology is the classical one).

2.13. Proposition. *In a morphism of topoi* $\Phi : Sh(\mathfrak{B}) \longrightarrow Sh(\mathfrak{C})$*,* Φ^* *preserves weak equivalences, and thus it descends to the homotopy categories.* Φ_* *preserves fibrations and IWE's of fibrant objects, thus it defines a right derived functor (cf.* [Q]*, Ch.I.4)* $\mathbb{R}\Phi_* : \mathrm{Ho}(\mathfrak{B}) \longrightarrow \mathrm{Ho}(\mathfrak{C})$*, right adjoint to* Φ^**.*

2.14. Remark. (i) With the "localization functors" $\lambda : \mathbb{S}Sh\,(B) \longrightarrow \mathrm{Ho}\,(\mathfrak{B})$ understood where necessary, the right derived functor $\varepsilon : \Phi_* \to \mathbb{R}\Phi_*$, defined up to natural isomorphism, if it exists at all, is an initial object among pairs (τ, Θ) consisting of a functor $\Theta : \mathrm{Ho}\,(\mathfrak{B}) \longrightarrow \mathrm{Ho}\,(\mathfrak{C})$ and a transformation $\tau : \Phi_* \to \Theta$.

(ii) $\mathbb{R}\Phi_*\,(\mathcal{X})$ is isomorphic to $\Phi_*\,(\mathcal{X}')$, for any globally fibrant object $\mathcal{X}'$ equivalent to $\mathcal{X}$.

(iii) The proof below assumes that the sites have enough stalks, but the proposition holds without that, with a similar proof.

Proof. Stalks of $\mathfrak{B}$ become stalks of $\mathfrak{C}$ after composing with Φ , so the statements about Φ^* are clear. It follows from exactness that Φ^* preserves cofibrations (monomorphisms), and by adjointness it follows that Φ_* preserves fibrations, weakly equivalent fibrations, and takes weakly equivalent fibrant objects to weakly equivalent fibrant objects. The construction of $\mathbb{R}\Phi_*$, and its adjointness to Φ^*, follows as in Thm. 4.3 of [Q], Ch. I.4. ∎

2.15. Example. (i) Let $\mathfrak{B}$ be any site and $\mathfrak{C}$ the underlying category with the trivial topology, generated by taking only isomorphisms as coverings. $Sh\,(\mathfrak{C}) = \mathrm{Pre}\,(\mathfrak{B})$, the category of presheaves (contravariant functors). The identity $i : \mathfrak{C} \longrightarrow \mathfrak{B}$ induces a morphism of topoi $\mathrm{I} : Sh(\mathfrak{B}) \longrightarrow \mathrm{Pre}(\mathfrak{B})$; I^* is the sheafification functor. $\mathbb{R}\mathrm{I}_*$ is a simplicial version of sheaf cohomology: $(\mathbb{R}\mathrm{I}_*\mathcal{X})\,(U) = \mathrm{Ext}(U; \mathcal{X})$. Note that $\mathrm{I}^* \circ \mathrm{I}_* = \mathrm{Id}$, whence, deriving, $\mathrm{I}^* \circ \mathbb{R}\mathrm{I}_* = \mathrm{Id}$. So, any sheaf $\mathcal{X}$ is "the sheafification of its cohomology functor", $U \mapsto \mathrm{Ext}(U; \mathcal{X})$.

(ii) Let $\mathcal{X} \in Sh(\mathfrak{C})$, $\mathfrak{B} = \mathfrak{C}/\mathcal{X}$; the obvious functor $p : \mathfrak{B} \longrightarrow \mathfrak{C}$ verifies both (a) and (b), and we get a triple (Π_+, Π^*, Π_*): the forgetful functor, the product with $\mathcal{X}$, while the last sends a sheaf $\mathcal{F}$ on $\mathfrak{C}/\mathcal{X}$ to $c \mapsto \mathrm{Hom}(C \times \mathcal{X}; \mathcal{F})$. The first two derive trivially, but the third, $\mathbb{R}\Pi_*$, is more interesting, giving rise to the *internal mapping stack* functor $\mathcal{E}xt(\mathcal{X}, \mathcal{Y}) := \mathbb{R}\Pi_* \circ \Pi^*\mathcal{Y}$. It satisfies the adjointness relation $[\mathcal{Z}, \mathcal{E}xt(\mathcal{X}, \mathcal{Y})] = [\mathcal{Z} \times \mathcal{X}, \mathcal{Y}]$.

2.16. **Underlying topological space.** The following notions and arguments are directed toward an abstract approach to the definition of the "underlying topological space" of a stack. This approach was suggested by the second author, as is pointed out at the end of the paper [Si3] which gives a different approach. This occured before the appearance of a similar argument in Voevodsky [?] and Morel-Voevodsky [MV]; we apologize for the overlap.

2.17. Definition. Define a sheaf X over $\mathfrak{C}$ to be (cohomologically) *contractible* if it satisfies the following three conditions:

(i) for any constant sheaf of sets Y, $\mathrm{Hom}\,(X; Y) = Y$;

(ii) for all (constant) coefficient groups G, $H^1\,(X; G) = *$;

(iii) for all constant Abelian coefficient groups A, $H^q\,(X; A) = 0$ for $q > 0$.

$\mathfrak{C}$ is *locally contractible* if $Sh\,(\mathfrak{C})$ contains a generating subcategory of contractible objects.

The site of complex analytic spaces in the classical topology is the main example.

2.18. Proposition. *If* $\mathfrak{C}$ *is locally contractible, the constant sheaf functor from* $\mathbb{S}\mathfrak{Sets}$ *to* $\mathbb{S}Sh\,(\mathfrak{C})$ *has, on homotopy categories, a left adjoint* $\mathrm{holim}_{\,\mathfrak{C}\to} : \mathrm{Ho}\,(\mathfrak{C}) \longrightarrow \mathrm{Ho}$.

2.19. Remark. Let $\lim_{\mathfrak{C}\to}$ be the left adjoint of the constant sheaf functor on the *simplicial* categories. (Its existence only requires $\mathfrak{C}$ to be "locally connected", that is, carry a generating subcategory of sheaves verifying condition (i) above). Then, $\mathrm{holim}_{\mathfrak{C}\to}$ can be shown to be its left derived functor, in the sense of Quillen. When $\mathfrak{C}$ is given the trivial topology (in which coverings are generated by the isomorphisms), sheaves over $\mathfrak{C}$ are the same as presheaves, i.e. contravariant functors, and the two functors just defined reduce to the usual direct limit and the Bousfield-Kan homotopy direct limit over $\mathfrak{C}$, respectively. In general, both functors depend on the topology of $\mathfrak{C}$.

Proof. Let $\mathfrak{G} \subset \mathfrak{C}$ be a generating subcategory of contractible objects. Replacing contractible objects by points defines a functor $\chi : \mathbb{S}_K\mathfrak{G} \longrightarrow \mathbb{S}\mathfrak{Sets}$, left adjoint to the "constant sheaf" functor. From (i)–(iii), we get that $H^n\,(\chi(\mathcal{X}); A) = H^n\,(\mathcal{X}; A)$, where the coefficient group A (abelian, if $n > 1$) may be twisted by any 1-cocycle with constant (possibly non-abelian) coefficients.

Cohomology in $\mathbb{S}Sh\,(\mathfrak{C})$ depends only on the Illusie weak homotopy type, and Whitehead's theorem for simplicial sets implies that χ turns IWE's into (regular) weakequivalences. Therefore, χ descends to a functor from $\mathrm{Ho}\,(\mathfrak{C})$ to Ho. This is the desired $\mathrm{ho}\lim_{\mathfrak{C}\rightarrow}$. To see the adjointness property, consider hypercoverings $\mathfrak{U} \to \mathfrak{X}$ in $\mathbb{S}_K\mathfrak{G}$; we have the equalities, in which the Kan complex Z doubles as a constant simplicial sheaf over $\mathfrak{C}$,

$$[\chi(\mathfrak{X}); Z] = [\chi(\mathfrak{U}); Z] = \pi_0\mathrm{Hom}_\bullet\,(\chi(\mathfrak{U}); Z) = \pi_0\mathrm{Hom}_\bullet\,(\mathfrak{U}; Z) \tag{2.20}$$

By the hypercovering theorem (2.7), the limit over $\mathfrak{U}$'s of the right-hand term gives $[\mathfrak{X}; Z]$, as desired. ∎

2.22. Remark. Local contractibility (2.17) applies to the category of analytic spaces. In that case, the functor χ can also be represented by the "geometric realization" [M], Ch.III of the simplicial topological space underlying the analytic stack [Si3].

We close with the following observation. For a small category $\mathfrak{I}$, the product $\mathfrak{C} \times \mathfrak{I}$ has a natural topology, pulled back from $\mathfrak{C}$.

2.23. Proposition. *Homotopy direct limits and homotopy inverse limits in* $\mathbb{S}Sh\,(\mathfrak{C})$*, indexed by a small category* $\mathfrak{I}$*, are representable in* $\mathrm{Ho}\,(\mathfrak{C})$ *as the left and right adjoints of the "constant diagram" functor (pull-back along* $\mathfrak{I}$*)* $\mathrm{Ho}\,(\mathfrak{C}) \longrightarrow \mathrm{Ho}\,(\mathfrak{C} \times \mathfrak{I})$*.*

Proof. $\mathrm{holim}_{\mathfrak{I}\leftarrow}$ is simply the right derived direct image for the projection along $\mathfrak{I}$. For the direct limit $\mathrm{holim}_{\mathfrak{I}\rightarrow}$, one can use the explicit construction of [BK], Ch. XII.2.1 (with a reversal of arrows in $\mathfrak{I}$, because their diagrams are covariant). This is a geometric construction, so it can be performed stalkwise, and preserves IWEs. Adjointness is seen as follows. Choose $(\mathfrak{X}^i)_{i\in\mathfrak{I}}$ in $\mathbb{S}Sh\,(\mathfrak{C} \times \mathfrak{I})$ and a fibrant $\mathfrak{Y}$ in $\mathbb{S}Sh\,(\mathfrak{C})$; we must show that $\mathrm{Ext}^{\mathfrak{C}}(\mathrm{holim}_{\mathfrak{I}\rightarrow}\mathfrak{X}^i; \mathfrak{Y}) = \mathrm{Ext}^{\mathfrak{C}\times\mathfrak{I}}(\mathfrak{X}^i; \mathfrak{Y}^i)$, with $\mathfrak{Y}^i = \mathfrak{Y}$ for each i. By the assumption on $\mathfrak{Y}$ and the relation between homotopy limits in [BK], Ch.XII.4,

$$\begin{aligned}\mathrm{Ext}^{\mathfrak{C}}\left(\mathrm{holim}_{\mathfrak{I}\rightarrow}\mathfrak{X}^i; \mathfrak{Y}\right) &= \mathrm{Hom}_\bullet^{\mathfrak{C}}\left(\ \mathrm{holim}_{\mathfrak{I}\rightarrow}\mathfrak{X}^i; \mathfrak{Y}\right) = \\ &= \mathrm{holim}_{\mathfrak{I}\leftarrow}\mathrm{Hom}_\bullet^{\mathfrak{C}}\left(\mathfrak{X}^i; \mathfrak{Y}\right)\end{aligned} \tag{2.24}$$

Because $\mathfrak{Y}$ is fibrant in $\mathfrak{C}$, each $\mathrm{Hom}_\bullet^{\mathfrak{C}}(\mathfrak{X}^i; \mathfrak{Y})$ is a Kan complex, and then, by [BK], Ch. XI, $\mathrm{holim}_{\mathfrak{I}\leftarrow}\ \mathrm{Hom}_\bullet^{\mathfrak{C}}(\mathfrak{X}^i; \mathfrak{Y})$ equals $\mathbb{R}\mathrm{lim}_{\mathfrak{I}\leftarrow}\mathrm{Hom}_\bullet^{\mathfrak{C}}(\mathfrak{X}^i; \mathfrak{Y})$ in Ho. Now, for fixed $\mathfrak{X}$, the functor $Y \mapsto \mathrm{Hom}_\bullet^{\mathfrak{C}}(\mathfrak{X}^i; \mathfrak{Y})$ from $\mathbb{S}Sh\,(\mathfrak{C})$ to $\mathbb{S}Sh\,(\mathfrak{I})$ preserves fibrations and weak equivalences of fibrant objects, *if we use the Bousfield-Kan model structure on* $\mathbb{S}Sh\,(\mathfrak{I})$ (in which fibrations are local fibrations). This implies the natural isomorphisms of functors

$$\begin{aligned}\mathbb{R}\mathrm{lim}_{\mathfrak{I}\leftarrow} \circ \mathbb{R}\mathrm{Hom}_\bullet^{\mathfrak{C}}\left(\mathfrak{X}^i; -\right) &= \mathbb{R}\left\{\mathrm{lim}_{\mathfrak{I}\leftarrow} \circ \mathrm{Hom}_\bullet^{\mathfrak{C}}\left(\mathfrak{X}^i; -\right)\right\} = \\ &= \mathbb{R}\mathrm{Hom}_\bullet^{\mathfrak{C}\times\mathfrak{I}}\left(\mathfrak{X}^i; \mathfrak{Y}^i\right)\end{aligned} \tag{2.25}$$

which is the desired equivalence. ∎

2.26. **Schemes of infinite type.** In the sections which follow, we will use the site $\mathfrak{F}$ of schemes of finite type over $\mathbb{C}$. The finite-type restriction is crucial to certain parts of our argument. We can use the notion of morphism of topoi to compare this with the bigger site $\mathfrak{N}$ of noetherian schemes over $\mathbb{C}$. The inclusion

$$\varphi : \mathfrak{F} \to \mathfrak{N}$$

pulls back coverings to coverings, and, at the same time, preserves fiber products and coverings (see the comment following 2.10). It thus defines an adjoint triple $(\Phi_!, \Phi^*, \Phi_*)$, in which the adjoint pairs $(\Phi_!, \Phi^*) : Sh(\mathfrak{N}) \to Sh(\mathfrak{F})$ and $(\Phi^*, \Phi_*) : Sh(\mathfrak{F}) \to Sh(\mathfrak{N})$ are morphisms of topoi. (Such a triple is called in [SGA4] an *essential morphism of topoi*, going from $\mathfrak{F}$ to $\mathfrak{N}$; the prototype is an open embedding). $\Phi_!$ extends φ to stacks: a simplicial representative of a stack $\mathfrak{X}$ on $\mathfrak{F}$, whose components are disjoint unions of schemes of finite type, also defines a stack on $\mathfrak{N}$. In the other direction, starting from a stack $\mathfrak{Y}$ on $\mathfrak{N}$, the associated stack $\Phi^*(\mathfrak{Y})$ on $\mathfrak{F}$ is just the restriction of $\mathfrak{Y}$ to the subcategory of schemes of finite type.

The composition $\Phi^* \circ \Phi_!$ is the identity; however the composition in the other direction is not the identity. Thus $Sh(\mathfrak{F})$ can be considered in some sense as a "direct factor" of $Sh(\mathfrak{N})$, but one should think of $Sh(\mathfrak{N})$ as having more objects. A typical example of an extra object which doesn't come from $Sh(\mathfrak{F})$ is $Spec(A)$ where A is a local ring such as the localization of a $\mathbb{C}$-algebra of finite type at a maximal ideal. This object doesn't appear in $Sh(\mathfrak{F})$ and indeed $\Phi_*(Spec(A))$ is the ind-scheme corresponding to the formal completion of $Spec(A)$ at its maximal ideal.

3. De Rham's theorem

Let $\mathfrak{A}$ be the site of analytic spaces in the classical topology, $\mathfrak{F}_{\acute{e}t}$ or $\mathfrak{F}_{Zar}$ that of complex schemes of finite type, in the étale or Zariski topologies. Both are closed under finite fiber products. In this section, $\mathfrak{C}$ will be used wherever either $\mathfrak{A}$ or $\mathfrak{F}$ would do, and objects of $\mathfrak{C}$ will be called *spaces*. It is sometimes convenient to enlarge $\mathfrak{A}$ and $\mathfrak{F}$ to include *formal spaces*, which are locally the formal neighborhoods of a subspace within a space.

The functor $\mathrm{Red} : \mathfrak{C} \longrightarrow \mathfrak{C}$ associates to every space the underlying reduced subspace.

3.1. Proposition. Red *induces an adjoint triple of functors* $(\mathrm{Red}, dR, \delta)$ *on* $Sh(\mathfrak{C})$.

Proof. Note that $\mathrm{Red}(U \times_V U) = \mathrm{Red}(U) \times_{\mathrm{Red}(V)} \mathrm{Red}(U)$, if $U \hookrightarrow V$ is an open embedding in $\mathfrak{C}$ (or an étale morphism in $\mathfrak{F}$), so the functor Red satisfies the weak form of condition (a) in (2.11.iii), and the full condition (b) in the preciding paragraph. ∎

By definition, the sheaf $dR(X)$ satisfies $\mathrm{Hom}(U; dR(X)) = \mathrm{Hom}(\mathrm{Red}(U); X)$ for any space U. Thus, the over category $\mathfrak{C}/dR(X)$ — whose objects are, by definition, natural transformations from representable functors to $dR(X)$ — agrees with Grothendieck's (big) infinitesimal site of X of [Gr], the category of pairs (U, f) consisting of a space U and a morphism $f : \mathrm{Red}(U) \to X$. The object $\mathrm{dR}(X)$ was called the *de Rham stack* of X in [Si2]; it is a zero-stack, in the sense that its higher sheafified homotopy groups vanish.

There is a structural morphism $X \twoheadrightarrow dR(X)$, right adjoint to the natural inclusion $\mathrm{Red} \to \mathrm{Id}$. Formal smoothness of X translates into the condition that this morphism should be a covering in $\mathfrak{C}$. If so, we can describe $dR(X)$ concretely, as the quotient of X under the formal equivalence relation whose graph is the formal neighborhood of the diagonal in $X \times X$; this is the "classifying stack" of the formal groupoid $\mathrm{diag}(X)\,\hat{}\, \rightrightarrows X$.

Note that the definitions of infinitesimal site, infinitesimal neighborhood, and of formal smoothness make sense for sheaves. It follows again that $\mathcal{X}$ is formally smooth iff the morphism $\mathcal{X} \to dR(\mathcal{X})$ is a covering, in which case, again, $dR(\mathcal{X})$ is the quotient of $\mathrm{diag}(\mathcal{X})\,\hat{}\, \rightrightarrows \mathcal{X}$.

3.2. Example. Consider a *strict ind-space* X, the direct limit of functors represented by a family of spaces X_n, nested by closed embeddings $X_n \subseteq X_{n+1}$. More sensible for us is the sheafification $\mathcal{X}$ of X over $\mathfrak{C}$, which is the sheaf direct limit of the X_n. Then, $\mathcal{X}$ is formally smooth iff the ind-space X is smooth in the sense of Shafarevich [**?**], that is, if the formal ring at any point is isomorphic to the (completed) symmetric algebra on the cotangent space.

Embedding a more general space X into a formally smooth, relatively nilpotent sheaf Y (most naturally, a formal space) induces an isomorphism $dR(X) \to dR(Y)$, and $dR(Y)$ can then be presented as indicated.

3.3. Proposition. *The pair* (dR, δ) *derives to an adjoint pair* $(dR, \mathbb{R}\delta)$ *on* $\mathrm{Ho}(\mathfrak{C})$. ∎

The algebraic and analytic de Rham theorems assert that $H^\star(dR(X); \mathcal{O}) = H^\star(\chi(X); \mathbb{C})$, for any space X; we recall here the functor $\chi = \mathrm{holim}_{\mathfrak{C}_\rightarrow}$ of Prop.(2.18). To extend this result to stacks, we rewrite $H^n(dR(X); \mathcal{O})$ as $[dR(X); K(\mathcal{O}, n)]$. The Eilenberg-MacLane object $K(\mathcal{O}, n)$ is constructed from the sheaf $\mathcal{O}$ by the standard simplicial procedure for Abelian groups ([M], Ch. V), and the equality of the two groups can be proved as in [Q], Ch. II.5, or else from the hypercovering theorem (2.7). Proposition (3.3) ensures that $[dR(X); K(\mathcal{O}, n)] = [X; \mathbb{R}\delta K(\mathcal{O}, n)]$; but this is not helpful until we identify $\mathbb{R}\delta$. We can do so quite explicitly in the analytic category, where we shall see that $\mathbb{R}\delta K(\mathcal{O}, n) = K(\mathbb{C}, n)$.

Let the functor dis : $Sh(\mathfrak{A}) \longrightarrow Sh(\mathfrak{A})$ assign to any sheaf the constant sheaf of its global sections. Calling $\pi : \mathfrak{A} \to (point)$ the obvious morphism of topoi, dis $= \pi^* \circ \pi_*$. For a space, dis is the constant sheaf of its points. There is a natural transformation dis $\to \delta$, obtained as follows: $\pi^* \circ \pi_*$ is the right adjoint of $\pi^* \circ \pi_+$, where π_+ is the "connected component" functor, left adjoint to π^*. The obvious adjunction morphism $\mathrm{Id} \to \pi^* \circ \pi_+$, which assigns to every point of a space the connected component containing it, factors through $\mathrm{Id} \to dR$, giving thus a natural transformation $dR \to \pi^* \circ \pi_+$. (This is obvious locally: on a space, the connected component is constant on the infinitesimal neighborhood of each point). Its right adjoint is the desired transformation dis $\to \delta$. There is induced a transformation $\mathbb{R}\mathrm{dis} \to \mathbb{R}\delta$ of derived functors on the homotopy category $\mathrm{Ho}(\mathfrak{A})$. Of course the higher cohomology over all of $\mathfrak{A}$ (which admits the point as a final object) is trivial, so $\mathbb{R}\pi_* = \pi_*$, and then $\mathbb{R}\mathrm{dis} = \pi^* \circ \mathbb{R}\pi_* = \mathrm{dis}$; so $\mathbb{R}$dis simply acts as dis on each simplicial component.

3.4. Proposition. *(a) For a space X, $\mathbb{R}\delta(X) = \mathrm{dis}(X)$.*

(b) For any Lie group G, $\mathbb{R}\delta(BG) = B(\mathrm{dis}(G))$; while for an Abelian Lie group A, $\mathbb{R}\delta K(A,n) = K(\mathrm{dis}(A), n)$.

(c) For any stack $\mathcal{Y}$ whose sheafified set of components, π_0, is a space, and whose sheafified homotopy groups are Lie groups, $\mathbb{R}\delta(\mathcal{Y}) = \mathrm{dis}(\mathcal{Y})$.

3.5. Remark. When a stack $\mathcal{Y}$ as in (c) above is of simple type (meaning that all π_1–sheaves are abelian, and their action on higher sheafified homotopy is trivial), dis($\mathcal{Y}$) is the constant sheaf whose stalk is the complex homotopy type with the discrete groups underlying the $\pi_i(\mathcal{Y})$ as homotopy groups, and with Postnikov k -invariants $k^{n+2}(\mathrm{dis}(\mathcal{Y})) \in H^{n+2}\left(\mathrm{dis}(\mathcal{Y}^{\leq n}); \mathrm{dis}\pi_{n+1}\mathcal{Y}\right)$ equal to the images of the sheafified k-invariants of $\mathcal{Y}$ from $H^{n+2}\left(\mathcal{Y}^{\leq n}; \pi_{n+1}\mathcal{Y}\right)$, under the natural map to $H^{n+2}\left(dR \circ \mathbb{R}\delta(\mathcal{Y}^{\leq n}); \pi_{n+1}\mathcal{Y}\right)$. If $\mathcal{Y}$ is not of simple type, dis($\mathcal{Y}$) will be a locally constant sheaf associated, in an obvious way, to the action of $\pi_1\mathcal{Y}$ on the homotopy type with the above k-invariants.

3.6. Remark. It is not the case, in general, that dis $= \delta$.

Proof. For a contractible Stein manifold U, cohomology computed in the over category $\mathfrak{A}/U$ agrees with cohomology over the small classical site of U, and

$$\begin{aligned}\mathrm{Ext}(U; \mathbb{R}\delta K(A,n)) &= \mathrm{Ext}(dR(U); K(A,n) = \\ &= K(\mathrm{dis}(A), n) = \mathrm{Ext}(U; K(\mathrm{dis}(A), n))\end{aligned} \tag{3.7}$$

from the adjointness relations and the analytic de Rham theorem on U. The sheafifications of the two outer Ext's (rigidified to functors) equal the inner functors, and (3.7) gives the desired isomorphism $\mathbb{R}\delta K(A,n) \approxeq (\mathrm{dis}(A), n)$. The result extends to n–truncated stacks $\mathcal{Y}$ (stacks with vanishing sheafified homotopy groups above dimension n), by induction on n, by virtue of Prop. 1 in [Q], Ch. I.4 ($\mathbb{R}\delta$ and dis, as right derived functors of left exact functors, preserve fibre sequences). Finally, over the small site of U, $\mathcal{Y}$ is the (homotopy) inverse limit of its Postnikov tower, and both $\mathbb{R}\delta$ and dis commute with homotopy inverse limits, giving the general result. ∎

3.8. Corollary. *For any complex Lie group A, representing a sheaf $\mathcal{A}$ over $\mathfrak{A}$, and any analytic stack X, de Rham's theorem $H^n(dR(\mathcal{X}); \mathcal{A}) = H^n(\mathcal{X}; \mathrm{dis}(A))$ holds. (A must be abelian if $n > 1$). For stacks over the Zariski or étale sites of $\mathfrak{F}$, $H^n(dR(\mathcal{X}); \mathbb{C})$ is the complex cohomology of $\chi(\mathcal{X})$.*

Proof. Follows from Prop.(3.4), parts (a) and (b), and the hypercohomology spectral sequence for a simplicial representative of $\mathcal{X}$. We get $[dR(\mathcal{X}); \mathcal{Y}] = [\mathcal{X}, \mathrm{dis}(\mathcal{Y})]$, for any $\mathcal{Y}$ as in part (c) of Prop.(3.4). ∎

3.9. Corollary. *De Rham's theorem $\mathbb{H}^\star(\mathcal{X}; \Omega^\bullet) = H^\star(\mathcal{X}; \mathbb{C})$ holds for formally smooth analytic sheaves. Correspondingly, for algebraic ones, $\mathbb{H}^*_{Zar}(\mathcal{X}; \Omega^\bullet) = H^\star(\mathcal{X}^{an}; \mathbb{C})$.*

Proof. $\mathcal{X}$ is formally smooth iff it can be covered by formally smooth (formal) spaces. We claim that restriction to the full subcategory of formally smooth objects in the site $\mathfrak{A}/\mathcal{X}$ does not change sheaf cohomology over $\mathfrak{A}/\mathcal{X}$. However, on formally smooth spaces, $\mathbb{C} \to \Omega^\bullet$ is a quasi-isomorphism. The result on the algebraic site follows from the equality $\mathbb{H}^\star(\mathcal{X};\Omega^\bullet) = H^\star(\mathcal{X}^{an};\mathbb{C})$, for formally smooth formal scheme X of finite type [H]. ∎

In particular, the naive form of de Rham's theorem holds for formally smooth ind-varieties (misleadingly called smooth in [?]). This is not completely obvious: it can happen that $\mathcal{X}$ cannot be written (not even locally) as an increasing union of smooth varieties.

3.10. Example. The ind-group $G[\Sigma]$ of algebraic maps from an affine curve to a Lie group G was shown in [BL] to be formally smooth and reduced. Using affineness of $G[\Sigma]$, the theorem shows that the Lie algebra cohomology[4] $H^\star(\mathfrak{g}[\Sigma];\Gamma(G[\Sigma];\mathcal{O}))$ agrees with the complex cohomology of the topological space underlying $G[\Sigma]$. This proposition (with convoluted proof) was used in [T1] to derive a van Est spectral sequence for $G[\Sigma]$. We shall seein Sect. 4 that $G[\Sigma]$ is not truly smooth, in the sense that it cannot be obtained (even locally) as increasing union of smooth subvarieties.

4. Regular singularities and the Riemann-Hilbert correspondence

In order to treat degree one nonabelian cohomology on stacks which are made out of quasiprojective varieties, we need to introduce a notion of *regular singularities*. To illustrate the problem, note that even if X is a smooth quasiprojective variety, and if $G = GL(n)$, then the *mapping stack* $\mathcal{E}xt(X, BG)$ is the moduli stack of rank n vector bundles with integrable connection on X; this includes connections with irregular singularities at the boundary $\overline{X} - X$, and in particular we cannot hope to have a comparison theorem comparing this with the topological cohomology of X^{top}.

We adopt the following general definition. Suppose G is an algebraic Lie group and suppose $\mathcal{X}$ is a stack. If $\rho \in \mathrm{Ext}(dR(\mathcal{X}), BG)$ then we say that ρ has *regular singularities* if for every quasiprojective curve C (maybe we should use spec of a discrete valuation ring here ???) and every morphism $f : C \to \mathcal{X}$ (i.e. point in $\mathcal{X}(C)$) the pullback

$$f^*(\rho) : dR(C) \to BG,$$

which is a principal G-bundle with integrable connection on C, has regular singularities.

We first note that this definition coincides with the usual definition when $\mathcal{X}$ is a quasiprojective variety: a differential system on $\mathcal{X}$ has regular singularities if and only if its restriction to every curve has regular singularities [?].

Next we define the stack $\mathcal{E}xt^{reg}(dR(\mathcal{X}), BG)$ of regular-singular morphisms to be the full substack of $\mathcal{E}xt(dR(X), BG)$ consisting of those points $P : S \to Hom(X_{dR}, BG)$ such that for every closed point $s \in S$, the restriction $P(s)$ (which is a morphism $dR(X) \to BG$) has regular singularities in the above sense.

It is clear that this notion is compatible with pullbacks: if $X \to Y$ is a morphism of stacks then we obtain a pullback morphism

$$\mathcal{E}xt^{reg}(dR(X), BG) \to \mathcal{E}xt^{reg}(dR(Y), BG).$$

In particular, if X is a stack then any regular-singular cohomology class pulls back over any quasiprojective variety Y to a regular-singular class in the usual sense.

We should point out that the stack $\mathcal{E}xt^{reg}(dR(X), BG)$ does not generally have very nice properties, for example it is not an algebraic stack (see [N]).

Our goal in this subsection is to prove the following theorem.

4.1. Theorem. *If X is a smooth quasiprojective variety and G is any algebraic group, then the Riemann-Hilbert morphism*

$$\mathcal{E}xt^{reg}(dR(X), BG)^{an} \to \mathcal{E}xt(\mathrm{dR}(X)^{an}, BG^{an})$$

is an equivalence of stacks on $\mathfrak{A}$.

[4]The correct definition of this cohomology involves a completed Koszul complex, which is then isomorphic to the global sections of the de Rham complex of the group of maps.

Of course it is well-known that on $Spec(\mathbb{C})$-valued points, the above morphism is an equivalence. The problem is to treat the dependence on parameters. This type of question has already been treated e.g. in the papers of Nitsure and Sabbah [N], [NS], but to our knowledge the precise statement we need hasn't occured yet in the literature.

Our proof is in several stages. We start by treating $G = GL(n)$; and here we first prove that the functor is fully faithful (this is the less well-known part), then we explain the standard method for seeing that it is essentially surjective. At the end, a standard tannakian argument allows us to go to any algebraic group G. It doesn't seem easy, on the other hand, to do the previous steps of the proof directly for G-bundles with arbitrary G.

Until further mention, we set $G := GL(n)$.

For full faithfulness, the first problem is to get a hold of what it means to have an S-valued point in $\mathcal{E}xt^{reg}(dR(X), BG)^{an}$ for a complex analytic space S. Recall that if S is a complex analytic space, then a morphism $S \to F^{an}$ comes, locally on S in its analytic topology, from a pair of morphisms

$$S \to P^{an}, \quad P \to F$$

where P is a scheme of finite type.

We would like to show that the morphism

$$\mathcal{E}xt^{reg}(dR(X), BG)^{an} \to \mathcal{E}xt(\mathrm{dR}(X)^{an}, BG)$$

is fully faithful. Interpreting morphisms between two differential systems E, F as flat sections of the bundle $E^* \otimes F$, the problem of proving that the above morphism is fully faithful is the same as the following problem. Suppose we have a scheme of finite type P and a vector bundle with integrable connection (E, ∇) on $P \times X$ relative to P, provided with a section β over $\{x\} \times P$. Suppose that for every closed point $p \in P$ the corresponding connection has regular singularities. Suppose furthermore that S is a complex analytic space with a morphism $S \to P^{an}$. Suppose that the corresponding analytic connection on $X^{an} \times S$ admits a section agreeing with β over $\{x\} \times S$. Then for every $s \in S$ we would like to find a scheme of finite type Q mapping to P such that over $X \times Q$ the algebraic vector bundle admits a flat section, and such that there is a neighborhood S' of $s \in S$ such that $S' \to P^{an}$ lifts to a morphism $S' \to Q^{an}$.

To do this, choose a smooth compactification Z of X (with a divisor D at infinity), and choose an extension of E to a coherent sheaf (which we also denote by E) on $Z \times P$. Choose a locally free sheaf L and an injection of coherent sheaves $E \hookrightarrow L$ which is strict over $X \times P$. Let $L \to F$ be the quotient. We may assume that F is torsion-free (this corresponds to assuming that E is saturated) and in turn we can choose an injection $F \hookrightarrow M$ with M locally free—and again we can suppose that the injection is strict over $X \times P$. Thus E is represented as the kernel of a morphism of locally free sheaves $L \to M$ and this morphism is strict on $X \times P$.

The connection ∇ (defined on E) has poles along the divisor D.

For any n let Q_n denote the scheme such that a morphism $R \to Q_n$ corresponds to a morphism $R \to P$ plus a section of the pullback of $L(nD)$ over $R \times Z$, going to zero in $M(nD)$ and annihilated by ∇. We have morphisms $Q_n \to Q_{n+1}$.

The morphism $Q_n \to P$ is injective on the level of points. Indeed, given two flat sections which agree with β on $\{x\} \times P$, they agree on $X \times P$ and since they are (for the purposes of defining our functor) considered as sections of $L(nD)$, they agree on $Z \times P$.

The morphism $Q_n \to P$ is proper. Suppose C is a quasiprojective curve with a point $0 \in C$, and let $C' = C - \{0\}$. Suppose we have a morphism $C' \to Q_n$ such that the composed morphism $C' \to P$ extends to C. Then there is an extension to $C \to Q_n$. To prove this, note that by pulling back we have a coherent sheaf E_C on $C \times Z$, with connection ∇ regular on $C \times X$, and we have a flat section (of $E(nD)$) defined over $C' \times Z$. This section agrees with the given section β along $C' \times \{x\}$. Analytically, the section corresponds to a fixed vector in the corresponding representation of the fundamental group, and the subset of points where a given vector is fixed is a closed subset of the variety of representations. This implies that the vector extends to an analytic flat section on $C \times X$. Now Hartogs theorem implies that our section is an algebraic section of $L(nD)$ over $C \times Z$.

A proper injective morphism is a closed immersion. Thus the Q_n are closed subschemes of P.

Now let Q^{an}_∞ denote the closed analytic subvariety of P defined by the condition that the monodromy representation of the system of equations, should fix the vector β. Note that

$$Q^{an}_n \subset Q^{an}_\infty .$$

On the other hand, the underlying set of points of Q^{an}_∞ is the union of the underlying set of points of the Q_n. Furthermore, locally in the analytic topology this union stabilizes, the reason for which we now explain. We can stratify P by locally closed subvarieties P_i and over each one of these subvarieties, choose a locally free extension E_i of E to $Z \times P_i$ such that the extension ∇_i has logarithmic singularities (recall the hypothesis that over every point, ∇ had regular singularities). On each P_i, define a real-valued function which is the maximum (or minimum ???) of 0 and of the real parts of the eigenvalues of the residue of the extension ∇_i. Combine these together into a function we denote by φ on P. It is continuous on each P_i.

We claim that φ is locally bounded at any point p of P. To see this, note that in view of its definition (as the maximum of the real part of a finite number of multivalued algebraic functions) it suffices to prove that for an algebraic curve C passing through p and with $C - \{p\}$ contained in a single stratum P_i, the function is bounded along C. Furthermore, the residues in question can be measured by restricting to curves in Z which cut all of the components of the divisor D. Therefore for this claim we can assume that Z is a curve. Now our extension E_i is a bundle over $(C - \{p\}) \times Z$, but this bundle extends to a locally free sheaf on $C \times Z$ (since with all of our reductions, $C \times Z$ is a smooth surface). The connection ∇_i is logarithmic away from $\{p\} \times D$ but Hartog's theorem implies that it takes values in the logarithmic differentials everywhere. Thus we have a logarithmic connection on $C \times Z$ and in the definition of the function φ we obtain a function which extends continuously to C. Note that the value of this extension at the point p can differ from the value of φ because, p being in a different stratum P_j, the extension constructed in the present paragraph might differ from the extension E_j used to define $\varphi(p)$. However, existence of the continuous extension implies that φ is locally bounded.

Now a locally bounded function can be bounded above by a continuous function ψ. Add into ψ a function bounding the degree of poles in the gauge transformations relating the extensions E_i and E.

Now ψ gives a bound for the integer n necessary in expressing Q^{an}_∞ as a union of the Q^{an}_n. Namely, for any point $p \in Q^{an}_\infty$, we have that $p \in Q^{an}_n$ for all $n \geq \psi(n)$. This is because the poles of a flat section of a vector bundle with logarithmic connection are bounded by the real parts of the eigenvalues of the residue of the connection. In particular, we get that Q^{an}_∞ is locally in the analytic topology equal to an algebraic closed subset.

We have now obtained the answer to our problem for reduced complex analytic spaces S: given a morphism $S \to P^{an}$ such that the pullback vector bundle has an analytic flat section (agreeing with β), this means exactly that the morphism factors through Q^{an}_∞. Thus, locally on S in the analytic topology, it factors into one of the Q^{an}_n.

The above argument concerns only the set-theoretic structure of Q^{an}_∞, so it doesn't immediately apply to non-reduced spaces S. For this, though, a deformation-theoretic argument (???) shows that given an algebraic bundle with meromorphic connection and with a flat section, and given a deformation of the bundle together with its meromorphic connection, if the section deforms as a fixed vector of the monodromy representation, then it deforms as a flat section. (All of this in the case of a deformation over an artinian ring of finite length.)

Suppose now that we have a morphism $S \to P$ with a flat section agreeing with β, and factorization $S^{red} \to Q^{an}_n$. Locally on S, the nilpotent ideal defining S^{red} has finite order of nilpotence. The deformation-theory argument of the previous paragraph shows that any artinian scheme mapping into S,goes into Q_n. This is readily seen to imply that S maps into Q_n (indeed the functions in the ideal defining Q_n vanish on all artinian subschemes of S, which implies that they vanish on S).

All in all we have proven the following theorem. For this, note that a bundle with a flat connection and a nonvanishing section is a principal $\mathcal{P}$-bundle where $\mathcal{P} \subset GL(n)$ is the parabolic subgroup of matrices with zeros in the non-diagonal places of the first column. Then $\mathcal{E}xt^{reg}(dR(X), B\mathcal{P}) \to \mathcal{E}xt^{reg}(dR(X), BGL(n))$ represents the functor of vector bundles with regular integrable connection together with a nonvanishing flat section. Let $\mathcal{E}xt^{reg}(dR(X), BGL(n), \Gamma) \to \mathcal{E}xt^{reg}(dR(X), BGL(n))$ represent the functor of vector bundles with integrable connection together with a section which is allowed to vanish.

4.2. Theorem. *Suppose X is a smooth quasiprojective variety. Then the following diagram is cartesian.*

$$\begin{array}{ccc} \mathcal{E}xt^{reg}(dR(X), B\mathcal{P})^{an} & \to & \mathcal{E}xt(dR(X)^{an}, B\mathcal{P}^{an}) \\ \downarrow & & \downarrow \\ \mathcal{E}xt^{reg}(dR(X), BGL(n))^{an} & \to & \mathcal{E}xt(dR(X)^{an}, BGL(n)^{an}). \end{array}$$

This statement extends to sections which are not necessarily nonvanishing: the same diagram but with $Hom(dR(X), B\mathcal{P})$ replaced by $\mathcal{E}xt^{reg}(dR(X), BGL(n), \Gamma)$ is also cartesian.

∎

4.3. Corollary. *The morphism of 1-stacks*

$$\mathcal{E}xt^{reg}(dR(X), BGL(n))^{an} \to \mathcal{E}xt(dR(X)^{an}, BGL(n)^{an})$$

is fully faithful. Again this extends to full faithfulness of the morphism of stacks of vector bundles with integrable connections together with all (not necessarily invertible) morphisms between them (this latter is a 1 -stack which isn't a 1-stack of groupoids, and for which we don't have a notation yet!!!????).

Proof: Given two vector bundles with integrable connection E, F, a morphism between them may be seen as a flat section of $E^* \otimes F$; for this we can apply the previous theorem. ∎

We would now like to prove that the functor in the previous corollary is essentially surjective. For this, we need to be a bit more careful about the compactification.

Suppose X is a smooth quasiprojective variety, and choose a compactification $\overline{X}$ such that the complement $D := \overline{X} - X$ is a divisor with normal crossings. Let $dR(\overline{X}, logD)$ denote the formal category associated to the logarithmic de Rham complex of $(\overline{X}, D)$. Recall (???) that a *principal G-bundle with logarithmic connection* is a morphism $dR(\overline{X}, logD) \to BG$.

Recall that $G = GL(n)$. We claim that if $\rho : dR(X) \times S \to BG$ is an S-valued connection with regular singularities, then there exists a principal G-bundle on $\overline{X}$ with a connection having logarithmic singularities and restricting to ρ, after possibly localizing on the base S. In other words, we claim that the morphism

$$\mathcal{E}xt(dR(\overline{X}, logD), BG) \to \mathcal{E}xt^{reg}(dR(X), BG)$$

is essentially surjective (i.e. surjective on π_0).

In what follows we keep $G = GL(n)$. Also, we work uniquely in the analytic site for now (however, X is the analytic space associated to a quasiprojective variety so we still have the normal crossings compactification). Fix a real number α. We say that a representation $\rho : dR(\overline{X}, \log D) \to BG$ is *α -canonical* if the residues of the connexion along the components of D have eigenvalues whose real parts a satisfy $\alpha < a < \alpha + 1$. (Note in particular that a system which is α-canonical, has good eigenvalues in the terminology of Nitsure and Sabbah [NS].)

We say that a representation $\rho : dR(X) \to BG$ with regular singularities, is *non-α* if the arguments of the eigenvalues of the monodromy transformations around components of D are not equal to $exp(2\pi i\alpha)$.

Caution: The above notions are not stable under very many things. For example if one changes the normal crossings compactification then these notions change (the residue at the blow-up of a normal crossing is the sum of the two residues, so the condition on real parts is not in general preserved). Also, these notions are not preserved under tensor product. In particular it is not clear how to define similar notions for groups other than $GL(n)$ so for now we must restrict to $G = GL(n)$.

Let $\mathcal{E}xt^{\alpha-can}(dR(\overline{X}, \log D), BG)$ be the full substack of morphisms whose values (on closed points of the parametrizing scheme S) are α-canonical. Let $\mathcal{E}xt^{non-\alpha}(dR(X), BG)$ be the full substack of morphisms whose values are non-α.

4.4. Lemma. *Recall that here we work in the analytic category. The substacks*

$$\mathcal{E}xt^{non-\alpha}(dR(X), BG) \subset \mathcal{E}xt(\mathrm{dR}(X), BG)$$

and

$$\mathcal{E}xt^{\alpha-can}(dR(\overline{X}, \log D), BG) \subset \mathcal{E}xt(dR(\overline{X}, \log D), BG)$$

are open substacks. The morphism

$$\mathcal{E}xt^{\alpha-can}(dR(\overline{X}, \log D), BG) \to \mathcal{E}xt^{non-\alpha}(dR(X), BG)$$

is an equivalence.

Proof: ??? ∎

4.5. Lemma. *Consider X as an algebraic object, use the superscript an to denote the associated analytic space, and use subscripts to denote which site we are refering to. Recall also that $G = GL(n)$.*

The natural morphism

$$\mathcal{E}xt_{\mathfrak{Z}}(dR(\overline{X}, \log D), BG)^{an} \to \mathcal{E}xt_{\mathfrak{A}}(dR(\overline{X}, \log D)^{an}, BG)$$

is an equivalence of stacks. In particular, there is an analytic open substack

$$U_\alpha \subset \mathcal{E}xt_{\mathfrak{Z}}(dR(\overline{X}, \log D), BG)^{an}$$

such that the natural morphism

$$U_\alpha \to \mathcal{E}xt_{\mathfrak{A}}(dR(X)^{an}, BG)$$

is an equivalence onto the full open substack $\mathcal{E}xt_{\mathfrak{A}}^{non-\alpha}(dR(X)^{an}, BG)$.

Proof: The equivalence is a consequence, for example, of the arguments given in [Si1]. ∎

4.6. Corollary. *The morphism*

$$Hom_{\mathfrak{Z}}(dR(X), BG)^{an} \to Hom_{\mathfrak{A}}(dR(X)^{an}, BG)$$

is essentially surjective.

Proof: The open substacks of non-α objects cover $\mathcal{E}xt_{\mathfrak{A}}(dR(X)^{an}, BG)$. We have exhibited open subsets U_α of the analytification of $\mathcal{E}xt_{\mathfrak{Z}}(dR(\overline{X}, \log D), BG)$ which surject on the level of objects, onto the open substacks in this covering. These surjections factor through the analytification of $\mathcal{E}xt_{\mathfrak{Z}}(dR(X), BG)$, proving the surjection in question. ∎

We have now completed the proof of Theorem 4.1 for $G = GL(n)$. Note also that the proof of full faithfulness works for morphisms which are not necessarily invertible. It is clear that the functor in Theorem 4.1 (extended to the stacks which include all not necessarily invertible morphisms) is a tensor functor (the stacks in question being stacks of tensor categories). Standard tannakian considerations now imply Theorem 4.1 for any algebraic group G. (??? more details ???).

4.7. **The Riemann-Hilbert correspondence for simplicial presheaves:** Turn now to a general stack (simplicial presheaf) X on $\mathfrak{F}$. If G is an algebraic group, a morphism $dR(X) \to BG$ is *regular singular* if, for every smooth quasiprojective curve C and morphism $C \to X$, the pullback morphism $dR(C) \to BG$ is regular singular. Let $\mathcal{E}xt^{reg}(dR(X), BG)$ denote the stack of regular singular morphisms. As above, this means the stack whose Y-valued points ($Y \in \mathfrak{F}$) are the morphisms $Y \times \mathrm{dR}(X) \to BG$ such that for every closed point $y \in Y$ the associated morphism is regular singular in the above sense. We obtain the following theorem.

4.8. Theorem. *In the situation of the above paragraph, the morphism*

$$\mathcal{E}xt^{reg}(dR(X), BG)^{an} \to \mathcal{E}xt(\ \mathrm{dR}(X)^{an}, BG^{an})$$

is an equivalence of stacks on $\mathfrak{A}$.

Proof: We know this statement for smooth quasiprojective varieties X . We can resolve any Y (for the ??? topology ???) by a simplicial ind-scheme whose components are disjoint unions of smooth quasiprojective varieties; the statement then follows from 4.1. ∎

??????? Generalizing the above result on de Rham's theorem for formally smooth ind-varieties, we have:

4.9. Lemma. *If X is a formally smooth ind-variety, then a morphism $dR(X) \to BG$ is the same thing as a G-torsor over X with integrable connection (where the notion of connection is defined in the usual way with respect to the algebra of differential forms $\Omega^{\cdot}_X$).*

Proof ????

5. Mixed Hodge structure on cohomology

Deligne [D2] defines a *mixed Hodge structure* (mHs) on the hypercohomologies $\mathbb{H}^n(X_\bullet;\mathbb{Z})$ of a simplicial variety $X_\bullet$, functorial for simplicial maps. We refer to those papers (and numerous other expositions, Brylinski's article in [Br], [Hu]) for the basic definitions and simplicial constructions.

Call $\mathbb{M}$ the category of $\mathbb{Z}$-mixed Hodge structures (other ground rings could be used). Our first goal is to show that the functor $\mathbb{H}^n : \mathbb{S}\mathfrak{F} \to \mathbb{M}$ factors through the homotopy category $\mathrm{Ho}(\mathfrak{F})$ of stacks. This seems easy enough, in view of the following facts: first, a morphism in $\mathbb{M}$ that is an isomorphism of underlying $\mathbb{Z}$-modules is an isomorphism in $\mathbb{M}$; second, weak equivalences of stacks induce isomorphisms in (analytic) $\mathbb{Z}$-cohomology; and finally, inverting weak equivalences in $\mathbb{S}_K\mathfrak{F}$ turns it into the homotopy category $\mathrm{Ho}(\mathfrak{F})$ (2.7). However, we want no finiteness assumptions on our Hodge structures (as in [D2]), since our simplicial varieties may have, levelwise, infiniteley many connected components. This is the case, for instance, in any presentation of the stack of vector bundles over a variety.

This requires a minor deviation from the standard treatment. Note, for instance, that $\mathbb{H}^\star(X_\bullet;\mathbb{C}) \neq \mathbb{H}^\star(X_\bullet;\mathbb{Z}) \otimes_\mathbb{Z} \mathbb{C}$ if $X_\bullet$ has infinite type, whereas this always holds for *homology.* We get to the Hodge structure on homology through the *distributional de Rham complex.* On a compact $2n$-dimensional manifold X, this is $\mathcal{D}_X^{-2n} \xrightarrow{d} \mathcal{D}_X^{-2n+1} \xrightarrow{d} \dots \xrightarrow{d} \mathcal{D}_X^0$, where $\mathcal{D}_X^m$ are the distributions with values in $(2n-m)$-forms (thus, $\mathcal{D}_X^0$ is dual to the smooth functions on X). This differential graded coalgebra is covariant for C^∞ maps and resolves the homology of X. In the compact Kähler case, the Hodge bigrading and the classical collapse of the Hodge-to-de Rham sequence allow one to define the (pure) Hodge structure on $H_\star(X;\mathbb{C})$. For open but compactifiable X, we consider the *distributional log de Rham complex* $\mathcal{D}^\bullet_{\overline{X}}(-\log D)$, defined after chosoing a smooth Kähler compactification $\overline{X}$, with normal-crossing boundary divisor D (call this a *good compactification*). Recall that the smooth log de Rham complex $\mathcal{A}^{p,q}_{\overline{X}}(\log D)$ is $\mathcal{A}^{0,q}_{\overline{X}} \otimes_{\mathcal{O}} \Omega^p_{\overline{X}}(\log D)$; then, $\mathcal{D}^{p,q}_{\overline{X}}(-\log D)$ is the topological dual of $\mathcal{A}^{n-p,n-q}_{\overline{X}}(\log D)$. Poincaré duality ensures that this resolves the homology of X with its Deligne-Hodge structure. The inverse limit of $\mathcal{D}^{p,q}_{\overline{X}}(-\log D)$, over all good compactifications, is functorial in X alone, and defines a *mixed homological Hodge complex.* (The definition is as in [D2], §6, with the reversal of arrows, filtrations, and abandoning the requirement that the homologies be finite $\mathbb{Z}$-modules).

(b?) Beilinson's theorem that $\mathbb{D}$MHS = Deligne's derived category of mixed Hodge complexes: Check it for direct limits of MHS? Polarizable structures

6. Hodge-to-de Rham

Let us start with a refresher. The holomorphic (algebraic) de Rham complex $X \mapsto (\Omega^\bullet(X), d)$ is a sheaf over $\mathfrak{A}$ (respectively over $\mathfrak{F}$, with either the Zariski or étale topologies). Restricted to the small analytic site of a smooth space X, the complex resolves the constant sheaf $\mathbb{C}$. The "stupid" filtration on $(\Omega^\bullet, d)$,

$$F^p(\Omega^\bullet, d) := 0 \to 0 \to ... \to \Omega^p \xrightarrow{d} \Omega^{p+1} \to ... \tag{6.1}$$

leads to the *Hodge-to-de Rham* (or *Frölicher*) spectral sequence, converging to $H^\star(X;\mathbb{C})$. For proper algebraic X, the algebraic and analytic sequences can be shown to agree (using GAGA in the projective case, and the method of [D1] in general). The two sequences can differ in general, although their abutments agree.

6.2. **The naive Hodge-to-de Rham sequence.** Over singular analytic spaces, $(\Omega^\bullet, d)$ need not resolve the constant sheaf, and one substitute is the following. Embed X in some formally smooth formal analytic space $\mathfrak{X}$, with defining ideal $\mathfrak{I}$. It is a classical fact (see e.g. [H]) that de Rham's complex of $\mathfrak{X}$ resolves $\mathbb{C}$, so the "stupid" filtration defines a spectral sequence converging to $H^\star(X;\mathbb{C})$. This depends on $\mathfrak{X}$, but we may remedy that as follows. Define the *naive Hodge filtration* on $\Omega^\bullet(\mathfrak{X})$ by

$$N^p\Omega^\bullet(\mathfrak{X}) = \mathfrak{I}^p \xrightarrow{d} \mathfrak{I}^{p-1}\Omega^1(\mathfrak{X}) \xrightarrow{d} \mathfrak{I}^{p-2}\Omega^2(\mathfrak{X}) \xrightarrow{d} ... \tag{6.3}$$

convening that $\mathfrak{I}^k = \mathcal{O}$ if $k < 0$. Each $\mathrm{Gr}_N^p\Omega^\bullet(\mathfrak{X})$ is naturally a bounded complex of coherent $\mathcal{O}$-modules over X, and is quasi-isomorphic to $\Omega^p(X)[-p]$, if X is smooth, or even a formally smooth formal space. A diagonal embedding argument shows that the class of $\mathrm{Gr}_N^p\Omega^\bullet(\mathfrak{X})$ in the derived category $\mathbb{D}^b_{coh}(X)$ of coherent sheaves on X is independent of the choice of $\mathfrak{X}$; we shall denote it $\mathrm{Gr}_N^p\Omega^\bullet(X)$. We get a spectral sequence, depending only on X,

$$E_1^{p,q} = H^{p+q}\left(X; \mathrm{Gr}_N^p\Omega^\bullet(X)\right) \Longrightarrow H^{p+q}(X;\mathbb{C}) \tag{6.4}$$

This we can take as our extension of the Hodge-to-de Rham spectral sequence to arbitrary analytic spaces.

6.5. Remark. Feigin and Tsygan [FT] considered this construction in relation to the cyclic homology of affine schemes; they called N^p the Hodge filtration. Our qualifier "naive" underlines the fact (see e.g. [W], §5) that N^p does *not* always induce Deligne's Hodge filtration on $H^\star(X;\mathbb{C})$, when X is singular.

There is a more intrinsic description of the spectral sequence. If X is formally smooth, the formal groupoid $\mathrm{diag}(X)\hat{} \rightrightarrows X$ representing $dR(X)$ can be degenerated to its normal cone about X (here, X is identified with $\mathrm{diag}(X) \rightrightarrows X$). This normal cone is the *Dolbeault stack of* X [Si2]; it is the classifying stack of the formal neighborhood of the zero-section in the tangent bundle of X, viewed as a formal abelian group scheme over X. This degeneration filters $\mathcal{O}$ on $dR(X)$, and the spectral sequence (6.4) is associated to this filtration. (If X is not smooth, embed it first in a formally smooth $\mathfrak{X}$, and again degenerate $\mathrm{diag}(\mathfrak{X})\hat{} \rightrightarrows \mathfrak{X}$ to the normal cone about $X \subset \mathfrak{X}$). The resulting Dolbeault stack is independent of the choice of $\mathfrak{X}$ (again, by a diagonal embeding argument).

6.6. Remark. When X is reduced, the spectral sequence arises from the filtration of the structure sheaf $\mathcal{O}$ over Grothendieck's infinitesimal site $\mathfrak{C}/dR(X)$ of X (see the paragraph following Prop. 3.1) by powers of the nilradical.

6.7. Proposition. *The spectral sequence (6.4) is functorially defined on the homotopy categories of analytic, étale or Zariski stacks (it depends only on the weak equivalence class of simplicial objects).*

Proof. The $N^p\Omega^\bullet$ and $\mathrm{Gr}_N^p\Omega^\bullet$ can be rigidified to a sheaf of complexes over $\mathfrak{F}$, as follows: the category of smooth formal thickenings $X \subseteq \mathfrak{X}$, with arrows $f : (X \subseteq \mathfrak{X}') \to (X \subseteq \mathfrak{X})$ being the formally smooth morphisms extending the identity on X, is a left directed system. The $\mathrm{Gr}_N^p\Omega^\bullet(\mathfrak{X})$ are contravariant, and their direct limit complexes are the desired sheaves over $\mathfrak{F}$ (or $\mathfrak{A}$). ∎

6.8. Example. With the exception of the "fake smooth" ind-varieties in the final subsection, the only examples we have naturally encountered are smooth. Not surprisingly, the Hodge cohomolgies admit nice descriptions.

(a) The case of BG, for any complex Lie group G, was studied by Cathelineau , who identified the $E_1^{p,q}$ term $H^q(BG;\Omega^p)$ with the holomorphic group cohomology $H_G^{q-p}(\mathrm{Sym}^p\mathfrak{g}^t)$. (When G is algebraic, this is the same as algebraic group cohomology). In this light, the Hodge-to-de Rham sequence appears as a holomorphic analogue of the Bott-Shulman [?] spectral sequence for real Lie groups, $E_1^{p,q} = H_G^{q-p}(\mathrm{Sym}^p\mathfrak{g}^t)$ (involving continuous group cohomology). Cathelineau's identification involves the bar presentation of BG, and the identification of the bar complex on $\Lambda^\bullet\mathfrak{g}^t$ (invariant differential forms on G) with $\mathrm{Sym}^\bullet\mathfrak{g}^t$. The sequence collapses for reductive G, but not, say, for $\mathbb{G}_a$, where we get the Koszul complex on the dual of $\mathfrak{g}_a$.

(b) **Carlos: Add K(V,n)?**

(c) Another generalization of (a) is the stack X/G (X smooth). An argument akin to Cathelineau's identifies $E_1^{p,q} = H^q(X/G;\Omega^p)$ with the equivariant holomorphic hypercohomology $\mathbb{H}_G^q(X;\mathrm{Gr}^p_{\mathrm{Hodge}})$ of a complex $\mathrm{Gr}^p_{\mathrm{Hodge}} = \bigoplus_{r+s=p}\mathrm{Sym}^r\mathfrak{g}^t\otimes\Omega^s$ of vector bundles on X, with differential $\sum_a \xi^a\otimes\iota(\xi_a)$. (Here, ξ_a is a basis of $\mathfrak{g}$ and ξ^a the dual basis of $\mathfrak{g}^t$; $\iota(\xi_a)$ is the interior multiplication on $\Omega^\bullet$ by the vector field defined by ξ_a). For reductive G and compact, Kähler X, the sequence collapses at E_1: indeed, if G is reductive (and connected, for simplicity),

$$H_G^\bullet(X;\mathrm{Sym}^\bullet\mathfrak{g}^t\otimes\Omega^\bullet) = \left(\mathrm{Sym}^\bullet\mathfrak{g}^t\right)^G\otimes H^\bullet(X;\mathbb{C}),$$

with total degrees matching on both sides. The right-hand side is already (additively) the equivariant complex cohomology $\mathbb{H}_G^\star(X;\mathbb{C})$, by a theorem of Deligne's [D1]; so no further differentials are induced by $\sum_a \xi^a\otimes\iota(\xi_a)$, nor can there be any higher Hodge-to-de Rham differentials.

(d) In [T2], §7, the collapsing result is generalized to the substack X^{ss}/G, where X^{ss} is the open set of G-semi-stable points on a (polarized) projective variety X. This is then used to prove the collaps of the Hodge-to-de Rham sequence of the moduli stack $\mathfrak{M}$ of G -bundles over a smooth proper curve. Note that $\mathfrak{M}$ is a smooth 1-stack and, while it is not proper, it satisfies the completeness part of the valuative criterion of properness. We do not have analogous results for bundles over higher-dimensional varieties; note that those stacks are neither smooth, nor complete in the valuative sense, so the case of curves may be a fortunate accident.

6.9. **The DuBois complex.** Following a conjecture of Deligne ([D2], §9), the "correct" extension of $\Omega^\bullet$ and its Hodge filtration to singular spaces was constructed by DuBois [Du]. Over any variety X, one gets a resolution of the constant sheaf $\mathbb{C}$ by a complex $(\underline{\underline{\Omega}}^\bullet, d)$ whose terms are $\mathcal{O}$-modules and whose arrows are first-order differential operators, as follows: choose a *smooth simplicial resolution* $\varepsilon : X_\bullet \to X$ and set $(\underline{\underline{\Omega}}^\bullet, d) := \mathbb{R}\varepsilon_*(\Omega^\bullet(X_\bullet), d)$, with the derivation $\mathbb{R}$ done in the Zariski (or étale) topology. Here, a "smooth resolution" is a hypercovering in the topology on $\mathfrak{F}$ generated by proper surjective maps, in which, additionally, all the X_n are smooth. This complex carries a decreasing *Hodge filtration* F, inherited from the naive Hodge filtration on $\Omega^\bullet(X_\bullet)$. DuBois shows that the triple $(\underline{\underline{\Omega}}^\bullet, d, F)$ is well-defined up to filtered quasi-isomorphisms, independent of the resolution. Consequently, the associated graded complexes $\underline{\underline{\Omega}}^p_X := {\mathrm{Gr}^p}_{\mathrm{Hodge}}(\underline{\underline{\Omega}}^\bullet, d) \simeq \mathbb{R}\varepsilon_*\Omega^p(X_\bullet)$ are unique in the derive category $\mathbb{D}^+_{coh}(X)$ (up to canonical isomorphism). Finally, when X is proper, the spectral sequence associated to the Hodge filtration of collapses at E_1, and yields Deligne's Hodge filtration on $H^\star(X;\mathbb{C})$.

6.10. Remark. (a) DuBois requires the simplicial maps in $X_\bullet$ themselves to be proper, but this is not truly restrictive, as these more conservative resolutions are cofinal among the liberal ones (cf. the construction in [D2], §5).

(b) On the analytic site, one also requires that $X_\bullet$ should be algebraic over X.

(c)The $\mathrm{Gr}^p_N\Omega^\bullet(X)$ defined earlier maps naturally to $\underline{\underline{\Omega}}^p(X)$, but does not generally agree with it.

To talk about $H^\star(\mathfrak{X};\underline{\underline{\Omega}}^p)$ for an $\mathfrak{A}$-stack $\mathfrak{X}$, we must define $(\underline{\underline{\Omega}}^\bullet, d, F)$ as a filtered complex of sheaves over $\mathfrak{A}$. It is not so easy to rigidify $\underline{\underline{\Omega}}^\bullet$, because the category of resolutions of X is not left filtering (let equalizers exist only up to simplicial homotopy, cf. Thm.2.7). Instead, we rephrase DuBois' construction more abstractly. Let $\mathfrak{A}_{psc}$ be the site of analytic spaces in the Grothendieck topology generated by proper-surjective maps and by classical open coverings (similarly, use étale coverings to define $\mathfrak{F}_{psc}$); this means that a sheaf in the psc-topology is a presheaf which satisfies the sheaf conditions (F.1), (F.2) of [G] both for proper-surjetive coverings and for classical coverings. (By Lemma (1.3) in *loc. cit.*, this determines the psc-topology). There is an obvious morphism of topoi $\Phi : \mathfrak{A}_{psc} \longrightarrow \mathfrak{A}$.

6.11. Proposition. *$(\underline{\underline{\Omega}}^\bullet, d, F)$ is filtered quasi-isomorphic to the restriction of $\mathbb{R}\Phi_*(\Omega^\bullet, d, F)$ to the small site of X.*

Proof. A natural arrow $(\underline{\underline{\Omega}}^\bullet, d, F) \to \mathbb{R}\Phi_*(\Omega^\bullet, d, F)|_X$ is constructed as follows. A resolution $\varepsilon : X_\bullet \to X$ is a psc-hypercovering, and thus induces an isomorphism $\mathbb{R}\Phi_*(\Omega^\bullet, d, F)(X) \simeq \mathbb{R}\varepsilon_* \circ \mathbb{R}\Phi_*(\Omega^\bullet, d, F)|_{X_\bullet}$ in the derived category of filtered complexes over X. This, combined with the natural arrow $\mathbb{R}\varepsilon_*(\Omega^\bullet, d, F) \to \mathbb{R}\varepsilon_* \circ \mathbb{R}\Phi_*(\Omega^\bullet, d, F)|_{X_\bullet}$, gives our arrow. The agreement of $\underline{\underline{\Omega}}^p$ and $\mathbb{R}\Phi_*(\Omega^p)$ can now be checked on the cohomology sheaves. The result would immediately follow from the hypercovering theorem, if smooth resolutions were cofinal among psc-hypervocerings. This is not quite the case, but the following lemma shows that, concerning complexes bounded below, resolutions are "sufficiently cofinal": a large skeleton of $X_\bullet$ suffices to compute the cohomology in a fixed degree, and each skeleton is a successive composition of alternate classical and proper-surjective contractible local fibrations, neither of which spoil the computation of $H^q(\Omega^p)$. ∎

6.12. Lemma. *Given a hypercovering $X_\bullet \to X$ in the psc topology and an integer $k \geq 0$, there is a sequence $X^N_\bullet \to X^{N-1}_\bullet \to \ldots \to X^1_\bullet \to X^0_\bullet$ of relatively contractible Kan fibrations, alternatively for the proper-surjective and classical topologies, in which $X^0_\bullet$ is the constant simplicial variety X, each X^i_j, $i > 0$, is smooth, and such that the k-skeleton of $X^N_\bullet$ factors through that of $X_\bullet$.*

Proof. This follows form the fact that any psc covering is the composition of an alternating sequence of proper-surjective and classical coverings, and from Verdier's method to produce hypercoverings, as explained e.g. in [D2], §5. ∎

6.13. Proposition. *The DuBois cohomologies $H^\star(\mathfrak{X};\underline{\underline{\Omega}}^p)$ and the DuBois spectral sequence $E_1^{p,q} = H^q(\mathfrak{X};\underline{\underline{\Omega}}^p) \Rightarrow H^{p+q}(\mathfrak{X};\mathbb{C})$ of a stack $\mathfrak{X}$ over $\mathfrak{F}$ or $\mathfrak{A}$ are well-defined. If $\mathfrak{X}$ can be represented by a smooth simplicial variety, $H^\star(\mathfrak{X};\underline{\underline{\Omega}}^p) = H^\star(\mathfrak{X};\mathrm{Gr}^p_N\Omega^\bullet)$. On the other hand, if $\mathfrak{X}$ has a simplicial representative that is, levelwise, a disjoint union of compact Kähler varieties, $E_1 = E_\infty$ in the DuBois sequence.*

The collapse of the sequence follows from the existence of a smooth proper bi-simplicial hypercovering, and the collapse of Hodge-to-de Rham for the latter [D2].

6.14. **Application: Non-smoothnes of some ind-groups.** As an application, we get a collapsing result for the Hodge-to-de Rham sequence for ind-varieties. A non-trivial (we think) real-world consequence is indicated in Cor. (6.17) below, which arose in joint work of the second author with S. Fishel and I. Grojnowski [FGT].

6.15. Proposition. *Let X be a strict analytic ind-variety $X_n \subseteq X_{n+1} \subseteq X_{n+2} \subseteq ...$ (cf. Sect. 3), and assume:*

(a) ("Compactness") The X_n are projective algebraic varieties;

(b) ("Smoothness") Locally, near every point, X is equivalent to a direct limit of smooth analytic subvarieties.

Then, the Hodge-to-de Rham spectral sequence $E_1^{p,q} = H^q(X; \Omega^p) \Rightarrow H^{p+q}(X; \mathbb{C})$ collapses at E_1 and yields the Hodge filtration on $H^\star(X; \mathbb{C})$.

Indeed, smoothness ensures the agreement of the DuBois and naive de Rham complexes, and Prop.6.13 applies. (Note that $\Omega^\bullet$ has an obvious meaning whenever X is formaly smooth). There is a more general result here — unfortunately, of limited interest, as we don't have any convincing uses of the notions of smoothness and compactness for n-stacks, for $n \geq 1$.

6.16. Proposition. *Let the stack $\mathfrak{X}$ over $\mathfrak{F}_{étale}$ satisfy the conditions:*

(a) In the psc-topology, $\mathfrak{X}$ is equivalent to a simplicial space $X_\bullet$ which is, levelwise, a disjoint union of proper varieties;

(b) The analytified stack $\mathfrak{X}^{an}$ is representable by a smooth simplicial variety $Y_\bullet$.

Then, $H^q(\mathfrak{X}; \Omega^p) = H^{p,q}(\mathfrak{X}; \mathbb{C})$.

Proof. Condition (a) ensures the collapse of the algebraic and analytic DuBois sequences:

$$\mathbb{H}^q(\mathfrak{X}; \underline{\underline{\Omega}}^p) = \mathbb{H}^q(X_\bullet, \underline{\underline{\Omega}}^p) = \mathbb{H}^{p,q}(X_\bullet; \mathbb{C}) = H^{p,q}(\mathfrak{X}; \mathbb{C})$$

Part (b), on the other hand, implies that

$$\mathbb{H}^q(Y_\bullet, \Omega^p) = \mathbb{H}^q(Y_\bullet, \underline{\underline{\Omega}}^p) = \mathbb{H}^q(\mathfrak{X}; \underline{\underline{\Omega}}^p)$$

Since $H^{p,q}(\mathfrak{X}; \mathbb{C}) = H^{p,q}(Y_\bullet; \mathbb{C})$, we get the desired collapse. ∎

6.17. Corollary. *For a simple Lie group G and an affine curve Σ , the ind-group $G[\Sigma]$ of G-valued regular maps is not smooth, in the sense that it can not be realized locally as an increasing union of smooth complex analytic subvarieties. (Thus, it is not a complex Lie group).*

One should recall that $G[\Sigma]$ is formally smooth and reduced [BL], [T1]; there seems to be no naive way to measure its failure to be "genuinely" smooth.

Proof. Let $\Sigma = \mathbb{A}^1$; $G[\Sigma]/G$ is then a Zariski-open subset of the "basic flag variety" $X = G((z))/G[[z]]$ of the loop group of G. The latter is a union of the projective Schubert varieties, so it verifies condition (a) in Prop.(6.15). However, as we shall see in a moment, the Hodge-to-de Rham spectral sequence on X does not collapse at E_1; so (6.15.b) cannot hold. For general Σ, X is a principal $G[\Sigma]$-bundle, in the étale topology, over the smooth stack of G-bundles over the natural compactification of Σ; so (6.15 .b) would hold for X, if it did hold for $G[\Sigma]$.Regarding the failure of Hodge-to-de Rham collapse for X, observe that collapse at E_1 would require $H^1(X; \Omega^1) = H^2(X; \mathbb{C}) = \mathbb{C}$. Instead, it turns out that $H^1(X; \Omega^1) = \mathbb{C}[[z]]$. There is indeed a natural identification (see [FGT] for details) of $H^1(X; \Omega^1)$ with the continuous Lie algebra cohomology $H^1(\mathfrak{g}[[z]], \mathfrak{g}; \mathfrak{g}[[z]]dz)$ (with the adjoint action on the coefficients). Associated to $f \in \mathbb{C}[[z]]$, there is a Lie algebra 1-cocycle $\mathfrak{g}[[z]] \to \mathfrak{g}[[z]]dz$, to wit, $\gamma \mapsto f \cdot d\gamma$; and all these represent distinct classses in H^1. ∎

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