Coulomb branches for quaternionic representations

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Background

1. Pure topological gauge theory for a compact group G in 3D can be defined naïvely as the *sphere topology* of the classifying space BG.
2. It has a refinement in terms of the holomorphic Lagrangian geometry on an algebraic completely integrable system $\mathscr{C}$ (*Toda system*), with base the adjoint orbits $B := \mathfrak{g}_{\mathbb{C}}/G_{\mathbb{C}} = \operatorname{Spec} H^*(BG)$.
3. The space $\mathscr{C}$ is called the *Coulomb branch* for pure $3D$ gauge theory.
4. Some boundary conditions for this theory are $2D$ gauged A-models. *Compact* symplectic manifolds give (coherent sheaves supported on) holomorphic Lagrangians in $\mathscr{C}$, finite over the base.
5. The sections of these sheaves are their equivariant cohomologies; more generally, the spaces of states for the circle, made G-equivariant.
6. For instance, $\mathscr{C}$ itself is foliated by the flag varieties of G (ranging over all the small quantum parameters).
7. This can be viewed as a *character theory* for topological G-actions on linear categories, with $\mathscr{C}$ playing the role of conjugacy classes, in the sense of semiclassical calculus (irreducibles $\leftrightarrow$ Lagrangians).

Example

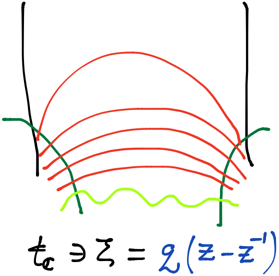
When $G = \mathrm{U}_1$, $\mathscr{C} = T^\vee\mathbb{C}^\times = \operatorname{Spec}\mathbb{C}[\tau, z^\pm] = \operatorname{Spec} H_*^{\mathrm{U}_1}(\Omega\mathrm{U}_1)$, with the Pontryagin product on $\Omega\mathrm{U}_1$.

(In general: $H_*^G(\Omega G)$, with G conjugating ΩG; a space first described by Bezrukavnikov-Finkelberg-Mirkovic.)

Then, $\mathbb{P}^1$ corresponds to the Lagrangians $\tau = q(z - z^{-1})$, the graphs of the differentials of $W(z) = q(z + z^{-1})$ in the toric mirror.

For SO_3, $\mathscr{C}(\mathrm{SO}_3)$ arises from $\mathscr{C}(\mathrm{U}_1)$ by dividing out the ± 1 Weyl action, blowing up $(\tau = 0, z = \pm 1)$, and deleting the proper transform of $\tau = 0$.

The $\mathbb{P}^1$-Lagrangians foliate the space, missing only the cotangent fibers over $z = \pm 1$ (the point flag variety).



Gauge theory "with matter"

For non-compact manifolds, the Lagrangians (4) need not be finite over B.

For example, the standard representation $\mathbb{C}$ of U_1 gives the Lagrangian $\{z = \tau\}$ in $T^\vee\mathbb{C}^\times$, with ring of functions $\mathbb{C}[\tau^\pm]$.
This is the equivariant *symplectic cohomology* of $\mathbb{C}$.

For a general complex representation V of G, one regularizes the symplectic cohomology by adding equivariance under scaling, with an equivariant *mass parameter* μ. (Here, μ just shifts τ, so is dispensable.)

Formally near $\mu = \infty$ (but not for finite μ) the Lagrangian is a section of B; the formula in terms of the weights $\{\nu\}$ of V is (in rank one)

$$z = \prod_\nu (\mu + \nu\tau)^\nu \quad \text{(Mirror formula)}$$

This infiniteness of the GLSM V, as boundary condition of gauge theory, seems to be remedied by introducing *matter fields* in the quaternionic representation $E := V \oplus \bar{V}$ in the $3D$ gauge theory.

Coulomb branches with matter

There is a Coulomb branch $\mathscr{C}(E)$ for more $\mathbb{H}$-representations E of G.

Gauge theory with matter assigns a vector space to S^2, which is an E_3 algebra for pictorial reasons (*sphere topology*), and $\mathscr{C}$ is Spec of that.

Based on substantial ideas from physics, Braverman, Finkelberg and Nakajima defined $\mathscr{C}(E)$ for the polarized case $V \oplus \bar{V}$.

In this lecture, we will

1. Recall the definition of $\mathscr{C}(V \oplus \bar{V})$
2. Give an explicit description in terms of $\mathscr{C}$ and the Mirror formula
3. Extend the definition to non-polarized representations
4. Give the Abelianization theorem for $\mathscr{C}$, allowing all computations.

Remark

The gauge theory should be recoverable from the Coulomb branch as its *Rozansky-Witten theory*. However, while this seems to work for $\mathscr{C}$, most Coulomb branches are singular, so the RW construction not clearly defined.

Acknowledgements

This is only a partial list, given my poor knowledge of the literature.

- The Coulomb branch for SU_2 was described by Seiberg and Witten
- Follow-up physics work for SU_n (Argyres et al.)
- The zero Coulomb branch was introduced by Bezrukavnikov, Finkelberg, Mirkovic in connection with the Toda system and Langlands duality (but not recognized as such at the time)
- Early computations on Coulomb branches by Hanany, Witten, et al.
- Braverman, Finkelberg, Nakajima first defined $\mathscr{C}(V \oplus \bar{V})$
- Bullimore, Dimofte, Gaiotto gave complete computations in the abelian case and conjectural constructions in general
- Recent work of Dimofte, Bullimore and others on the Physics side
- Sam Raskin explained to me a chiral homology construction of Coulomb branches and the mod 2 obstructions (due to Witten). This motivated the present construction.

Properties of Coulomb branches

1. The zero branch $\mathscr{C}$ is an abelian group scheme over the base B.
2. Each $\mathscr{C}(E)$ carries a Poisson structure (of degree (-2))
3. Each $\mathscr{C}(E)$ maps to B with Lagrangian fibers, and is free as a module.
4. Each $\mathscr{C}(E)$ carries a fiber-wise action of $\mathscr{C}$, and is birational to $\mathscr{C}$.
5. There are compatible multiplications $\mathscr{C}(E) \times_B \mathscr{C}(F) \to \mathscr{C}(E \oplus F)$.
6. $\mathscr{C}(V \oplus \bar{V})$ has two regular Lagrangian sections, corresponding to $V, \bar{V}$, whose ratio is given by the earlier Mirror formula.
7. *With the mass parameter included*, $\mathscr{C}(V \oplus \bar{V})$ is the quotient in affine schemes of two copies of $\mathscr{C}$ relatively translated by the mirror section.
8. Abelianization: $\mathscr{C}(G; E) = \mathscr{C}\left(T; E \ominus \mathfrak{g}_{\mathbb{C}}^{\oplus 2}\right)/W$, when the latter makes sense (e.g. the roots are weights of E but less is needed).

(3) and (8) $\Rightarrow$ all Coulomb branches from $\mathscr{C}$ and from SU_2, $V = \mathbb{C}^2$.

Pure topological gauge theory

Two naïve versions of gauge theory in $3D$ (related by Koszul duality) can be described homotopically. They fit in the framework of fully extended TQFT, but lose information.
Much like in $2D$, string topology for a compact manifold M can be generated by $C^*(M)$ or $C_*(\Omega M)$, we can use $C^*(M)$ and the E_2 algebra $C_*(\Omega^2 M)$ to define *sphere topology*, a partially defined $3D$ TQFT.
It is fully defined up to dimension 2; some $3D$ operations also exist (e.g. the E_3 structure on the space assigned to the sphere).

Here, $M = BG$ (so we can dispense with (co)chains). The space of states for S^2 is computed, in the two cases, as the E_2 Hochschild cohomology of $H^*(BG)$, resp. $H_*(\Omega G)$, giving formal completions of $\mathscr{C}$ along Lagrangians (the cotangent fiber at 1, or the zero-section, respectively, for the torus).

From the two naïve answers, the true answer can be guessed as $H^*_G(\Omega G)$. More generally, a closed surface Σ leads to the analogous equivariant homology of the stack of G-bundles. This gives a partial TQFT.

Wild speculation

This topological construction makes it seem that adding a representation E is irrelevant, because of its contractibility.
What shows up instead is a categorified version of the *virtual fundamental class* for an obstructed deformation problem. This does in fact carry a virtual fundamental class from the boundary theory, the GLSM of V.

It appears, on the level of surfaces, in the guise of a *constructible sheaf* on the stack Bun_G of G-bundles, which is (conjecturally) multiplicative for the TQFT gluing operations. A key contribution of [BFN] is a coefficient system on ΩG in which the sphere multiplication is strictly defined, bypassing the deformation and coherence checks.

The theory with matter should be generated by a conjectural *gauged holomorphic Fukaya 2-category of E*. For $V \oplus \bar{V}$, a simplified model could be the gauged endofunctor category of the Fukaya category of V. This seems difficult to define precisely.

The BFN construction

The construction uses the algebraic model $G_{\mathbb{C}}((z))/G_{\mathbb{C}}[\![z]\!]$ of ΩG, and the coefficient system produced is $G_{\mathbb{C}}[\![z]\!]$-equivariant.
The double coset $G_{\mathbb{C}}[\![z]\!]\backslash\Omega G$ represents the stack of bundles on the formal disk with doubled origin; it is multiplicative via the Hecke correspondence

$$\begin{array}{c} \{G_{\mathbb{C}}[\![z]\!] \backslash G_{\mathbb{C}}((z))/G_{\mathbb{C}}[\![z]\!]\} \times \{G_{\mathbb{C}}[\![z]\!] \backslash G_{\mathbb{C}}((z))/G_{\mathbb{C}}[\![z]\!]\} \\ \uparrow \\ G_{\mathbb{C}}[\![z]\!] \backslash G_{\mathbb{C}}((z)) \times_{G_{\mathbb{C}}[\![z]\!]} G_{\mathbb{C}}((z))/G_{\mathbb{C}}[\![z]\!] \\ \downarrow \\ G_{\mathbb{C}}[\![z]\!] \backslash G_{\mathbb{C}}((z))/G_{\mathbb{C}}[\![z]\!] \end{array}$$

One first produces an equivariant linear space $L_V \to \Omega G$, whose fibre-wise Borel-Moore homology defines the constructible sheaf $\mathscr{S}_V$.
Multiplicativity of $\mathscr{S}_V$ is implemented by a correspondence diagram on L_V, which matches the multiplication in the Hecke correspondence.

At a loop $\gamma \in G_{\mathbb{C}}(\!(z)\!)$, the fiber $L_V(\gamma)$ is the kernel of

$$V[\![z]\!] \oplus V[\![z]\!] \xrightarrow{\mathrm{Id}-\gamma} V(\!(z)\!)$$

which has finite codimension in $V[\![z]\!]$.
(This is H^0 of the V-bundle over the double-centered disk.)

We can dispense with the infinite-dimensional bundles by using $\mathrm{Bun}_G(\mathbb{P}^1)$ in guise of $G \setminus \Omega G$ and replace L_V with the total space of the *index sheaf*

$$\mathrm{Spec}\,\mathrm{Sym}\left[R\Gamma\left(\mathbb{P}^1; \mathscr{V}(-1)\right)^\vee\right],$$

where $\mathscr{V}$ is the universal bundle with fiber $\mathscr{V}$ on $\mathbb{P}^1 \times \mathrm{Bun}_G$.
The sheaf $\mathscr{S}_V \to \mathrm{Bun}_G$ is the fiber-wise compactly supported cohomology.
The "square root" of $\mathscr{S}$ that we will need in the non-polarized case is more transparent in this interpretation.

Remark

This last construction can be applied to any closed surface, but only yields a topological answer *after stabilization* (perturbation of $\bar\partial$, or high order poles to avoid Brill-Noether phenomena).

Quaternionic representations

When E has no invariant Lagrangian, the $3D$ theory has no obvious topological boundary conditions that would allow its reconstruction by a Mirror formula. (It may well have *conformal* boundary conditions.)

Instead, we will modify of the BFN construction of the constructible sheaf.

Note that using E in lieu of V produces a sheaf $\mathscr{S}_E$ for the doubled representation $E \oplus \bar{E}$. So what is needed is a square root of $\mathscr{S}_E$, in the following sense.

Each stratum of ΩG is shifted cohomologically by capping with the Euler class of H^0 of the universal bundle. Now, in K-theory, the Euler class of a vector bundle is its exterior algebra. A square root would be the Spin bundle, if one exists. This requires a reduction along Spin $\to$ SO $\to$ U of the structure group.

We will find something close enough to this for $H^0(E)$.

A Bott sequence (Wood's theorem)

This is the sequence, classified by $\eta : \Sigma^2 KO \to \Sigma^1 KO$,

$$KO \xrightarrow{\mathbb{C}\otimes_{\mathbb{R}}} KU \xrightarrow{\Omega^2(\mathbb{H}\otimes_{\mathbb{C}})} \Omega^2 BSp = \Sigma^2 KO \qquad \text{(P)}$$

whose relevance is seen when integrating E over $\mathbb{P}^1$,

$$E : BG \to BSp \quad \rightsquigarrow \quad R\Gamma(E(-1)) : \Omega G \to \Omega^2 BSp = \Sigma^2 KO :$$

it refines the complex structure of $R\Gamma$ to $\Sigma^2 KO$.
A polarization V of E would lift $R\Gamma(E)$ to $R\Gamma(V) \in KU$ in (P), and we could build $L_V, \mathscr{S}_V$ as before.
Now, an obstruction calculation shows that a lift in (P) *always* exists, just not an *additive* one. So the constructible sheaf $\mathscr{S}_V$ would seem not (E_2) multiplicative.

But that is not quite right.

Mod 2 obstruction to multiplicativity

By virtue of (P), additivity holds up to suspension by a real virtual vector bundle. Should this be oriented, the Thom isomorphism would remove the obstruction. (Similarly, a Spin structure would remove it in K-theory.)

The obstruction to lifting $B\mathrm{O}$ to $B\mathrm{Spin}$ consists of w_1, w_2, in the bottom of $B\mathrm{O}$; this is a spectrum W with $\pi_1 = \pi_2 = \mathbb{Z}/2$ connected by Sq^2. The obstruction to Spin reduction in (P) is then read from $\eta : \Sigma^2 KO \to \Sigma^1 W$. Since we need E_2 additivity in the reduction, the obstruction is

$$E : BG \to B\mathrm{Sp} \to \Sigma^3 W$$

and consists of a class in $H^4(BG; \mathbb{Z}/2)$ and, subject to its vanishing, a second one in $H^5(BG; \mathbb{Z}/2)$.

Remark

The argument needs fleshing out: one uses a lifting of ΩG in (P) to suspend the index sheaf and produce a linear space with a real structure.

Questions and Conjectures

1. The construction should work on an arbitrary curve and produce a vector space independent of the complex structure. TQFT would need a factorization rule for these, which must be spelt out. There will be no rigid construction as in [BFN].
2. Abelianization in higher genus should be a strong version of the Atiyah-Bott construction, splitting the homology. This may allow effective computation of the TQFT structure.
3. In the polarized case, the TQFT should admit the GLSM of V as a topological boundary theory.
4. Without polarization, conformal boundary conditions should exist. Can one construct the $3D$ TQFT from those, as one can from the topological boundary theory in the polarized case?

THANK YOU FOR LISTENING

Coulomb branch and the Langlands dual group $G^\vee$

Theorem (Bezrukavnikov-Finkelberg-Mirkovic + small improvements)

1. *Spec* $H_*^G(\Omega G)$ *is an algebraic symplectic manifold, isomorphic to the algebraic symplectic reduction* $T^*_{reg}G^\vee//_{Ad}G^\vee$.
2. *It is an affine resolution of singularities of* $(T^*T^\vee_{\mathbb{C}})/W$.
3. *The fiber of Spec* $H_*^G(\Omega G)$ *over* $0 \subset \mathfrak{t}_{\mathbb{C}}/W$ *is a Lagrangian submanifold* $\cong$ *Spec* $H_*(\Omega G)$. *("Free boundary condition".)*
4. *Spec* $K_*^G(\Omega G)$ *is is an algebraic symplectic orbifold, isomorphic to a twisted holomorphic symplectic reduction* $T^*_{reg}(LG_{\mathbb{C}})^\vee//_{Ad}(LG_{\mathbb{C}})^\vee$.
5. *It is an affine (orbi-)resolution of singularities of* $(T_{\mathbb{C}} \times T^\vee_{\mathbb{C}})/W$
6. *See (3), mutatis mutandis.*