Gauge theory and mirror symmetry

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Abstract. Outlined here is a description of *equivariance* in the world of 2-dimensional extended topological quantum field theories, under a topological action of compact Lie groups. In physics language, I am gauging the theories — coupling them to a principal bundle on the surface world-sheet. I describe the data needed to gauge the theory, as well as the computation of the gauged theory, the result of integrating over all bundles. The relevant theories are 'A-models', such as arise from the Gromov-Witten theory of a symplectic manifold with Hamiltonian group action, and the mathematical description starts with a group action on the generating category (the Fukaya category, in this example) which is factored through the topology of the group. Their mirror description involves holomorphic symplectic manifolds and Lagrangians related to the Langlands dual group. An application recovers the complex mirrors of flag varieties proposed by Rietsch.

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1. Introduction

This paper tells the story of equivariance, under a compact Lie group, in the higher algebra surrounding *topological quantum field theory* (TQFT). Speaking in riddles, if 2-dimensional TQFT is a higher analogue of cohomology (the reader may think of the Fukaya-Floer theory of a symplectic manifold as refining ordinary cohomology), my story of gauged TQFTs is the analogue of equivariant cohomology. The case of finite groups, well-studied in the literature [Tu], provides a useful and easy reference point, but the surprising features of the continuous case, such as the appearance of holomorphic symplectic spaces and Langlands duality, are missing there.

From another angle, this is a story of the categorified representation theory of a compact Lie group G, with the provision that representations are *topological*: the G-action (on a linear category) factors through the topology of G. One floor below, where the group acts on vector spaces, these would be not the ordinary complex representations of G, but the local systems of vector spaces on the classifying space BG. There is no distinction for a finite group, but in the connected case, BG is simply connected, and we must pass to the derived category to see

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anything interesting. The same will hold in the categorified story, where simply connected groups will appear to have trivial representation theory, before deriving. This observation suggests a straightforward homological algebra approach to the investigation, worthy of featuring as an example in a graduate textbook. Pursuing that road, however, leads to faulty predictions, even in the simplest case of pure gauge theory of a point (topological Yang-Mills theory). One reason for this failure is a curious predilection of interesting TQFTs to break the obvious $\mathbb{Z}$-grading information present, collapsing it to a $\mathbb{Z}/2$ grading, or encoding it in more labored form (as in the Euler field of Gromov-Witten theory [M]). The result is that homological algebra, which localizes the spectrum of a graded ring to its degree zero part, loses relevant information, which needs restoration by ulterior guesswork. In our example, we will see the homological information in the neighborhood of a Lagrangian within a certain holomorphic symplectic manifold, whereas most of the interesting 'physics' happens elsewhere.

The emerging geometric picture for this categorical topological representation theory is surprisingly attractive. Representations admit a character theory, but characters are now coherent sheaves on a manifold related to the conjugacy classes, instead of functions. The manifold in question, the *BFM space* of the Langlands dual Lie group $G^\vee$, introduced in [BFM], is closely related to the cotangent bundle to the space of conjugacy classes in the complex group $G^\vee_{\mathbb{C}}$. (For SU_2, it is the Atiyah-Hitchin manifold studied in detail in [AH].) Multiplicity spaces of G-invariant maps between linear representations are now replaced by multiplicity categories, whose 'dimensions' are the Hom-spaces in the category of coherent sheaves. (In interesting examples, they are the Frobenius algebras underlying 2-dimensional TQFTs.) There is a preferred family of simple representations, which in a sense exhausts the space of representations: they foliate the BFM space. Every such representation is 'symplectically induced' from a one-dimensional representation of a certain Levi subgroup of G: more precisely, it is the Fukaya category of a flag variety of G. This is formally similar to the Borel-Weil construction of irreducible representations of G by holomorphic induction. Recall that in that world there is another kind of "L^2-induction" from closed subgroups, which is right adjoint to the restriction functor. The counterpart of naïve induction also exists in our world, and gives the (curved) *string topologies* [CS] of the same flag varieties, instead of their Fukaya categories.

This story might seem a bit unhinged, were it not for the appearance of the governing structure in the work of Kapustin, Rozansky and Saulina [KRS]. Studied there are boundary conditions in the 3-dimensional TQFT associated to a holomorphic symplectic manifold X, known as Rozansky-Witten theory [RW]. Among those are holomorphic Lagrangian sub-manifolds of X, or more generally, sheaves of categories over such sub-manifolds. (The full 2-category of all boundary conditions does not yet have a precise definition.) The relation to gauge theory is summarized by the observation that gaugeable 2-dimensional field theories are topological boundary conditions for pure 3-dimensional topological gauge theory. The reader may illustrate this with an easy example: the representations of a finite group F are the boundary conditions for pure F-gauge theory in 2 dimensions; yet these representations are exactly the 1-dimensional topological field theories (vector spaces) which admit F-symmetry. Modulo the [KRS] description of Rozansky-Witten theory, my entire story is underpinned by the following

Meta-Statement. *Pure topological gauge theory in* 3 *dimensions for a compact Lie group* G *is equivalent to the Rozansky-Witten theory for the BFM space of the Langlands dual Lie group* $G^\vee$.

I shall offer no elucidation of this, beyond its inspirational value; however, strong indications of this statement have been known in the physics literature, at least for special G [SW, AF, MW]. Formulating this statement in a mathematically useable way will require an excursion through much preliminary material in §2-5. A small reward will come in §6, where we illustrate how these ideas can lead to 'real answers'.

A closing warning is that the results in this paper are partly experimental: enough examples have been checked to rule out plausible alternatives, but I do not claim to know proofs in full generality. In fact, the status of Floer-Fukaya theory makes such claims difficult to sustain, and the author has no special expertise on that topic. In topological cases, such as for string topology (Fukaya theory of cotangent bundles), precise statements and proofs are possible (and easy). More generally, the results apply to the abstract setting of differential graded (or A_∞-categories) with topological G-action, the question being to what extent the Fukaya category of a symplectic manifold with Hamiltonian G-action qualifies. (For non-compact manifolds, this depends on the 'wrapping' condition at ∞.) If nothing else, the paper can be read as a template for what a nice world should look like.

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2. Topological field theory

Topological field theory, introduced originally by Atiyah[A], Segal [S] and Witten [W], promised to systematize a slew of new 3-manifold invariants. The invariants of a 3-manifold M are thought to arise from *path integrals* over a space of maps from M to a target X. The latter is often a manifold, but in interesting cases, related to *gauge theory*, it is a stack. One example relevant for us will have X a holomorphic symplectic manifold, leading to *Rozansky-Witten theory* [RW]. The 2-dimensional version of this notion quickly found application to the counting of holomorphic curves, the Gromov-Witten invariants of a symplectic manifold X: these are controlled by a family of TQFTs parametrized by the even cohomology space $H^{ev}(X)$.

2.1. Extended TQFTs. Both theories above have a bearing on my story, once they are *extended down to points*. In the original definition, a d-dimensional TQFT is a symmetric, strongly monoidal functor form the category whose objects are closed $(d-1)$-manifolds and whose morphisms are compact d-bordisms, to the category $\mathfrak{Vect}$ of complex finite-dimensional vector spaces; the monoidal structures are disjoint union and tensor product, respectively. (Some tangential structure on manifolds is chosen, as part of the starting datum.) Fully extending the theory means extending this functor to one from the *bordism d-category $Bord_d$*, whose objects are points and whose k-morphisms are compact k-manifolds with corners (and some tangential structure), to some de-looping of the category of vector spaces: a symmetric monoidal d-category whose top three layers are complex numbers, vector spaces and linear categories, or a differential graded (dg) version of this. When $d = 2$, which most concerns us, the target is usually the 2-category $\mathfrak{LCat}$ of linear dg categories, linear functors and natural transformations. The reader may consult Lurie [L], references therein and the wide following it inspired, for a precise setting of higher categories.

Example 2.1 (2-dimensional gauge theory with finite gauge group F). This theory is defined for unoriented manifolds; among others, the functor Z_F which sends a point $*$ to the category $\mathrm{Rep}(F)$ of (finite-dimensional) linear representations of F, the half-circle bordism $\subset: \emptyset \to \{*, *'\}$ to the functor $\mathfrak{Vect} \to \mathfrak{Rep}(F) \otimes \mathfrak{Rep}(F)$ sending $\mathbb{C}$ to the (2-sided) regular representation of F, the opposite bordism $\supset: \{*, *'\} \to \emptyset$ to the functor $\mathfrak{Rep}(F) \otimes \mathfrak{Rep}(F) \to \mathfrak{Vect}$ sending $V \otimes W$ to the subspace of F-invariants therein. A closed surface gives a number, which is the (weighted) count of principal F-bundles. See for instance [FHLT] for a uniform construction of the complete functor and generalizations.

The first theorem of [L] is that an such extended TQFT $Z : Bord_d \to ??$ is determined by its value $Z(+)$ on the point, at least in the setting of *framed* manifolds. The object $Z(+)$, which we call the generator of Z, must satisfy some strong (*full dualizability*) conditions, but carries no additional structure, beyond being a member of an ambient d-category.

On the other hand, the ability to pass to surfaces with less structure than a framing on their tangent bundle forces additional structure on the generator $Z(+)$. The point (conceived together with an ambient germ of surface) carries a 2-framing, on which the group $\mathrm{O}(d)$ acts. Lurie's second theorem states that, given a tangential structure, encoded in a homomorphism $G \to \mathrm{O}(d)$, factoring the theory Z from $Bord_d$ through the category $Bord_d^G$ of d-folds with G-structure is equivalent to exhibiting $Z(+)$ as a fixed-point for the G-action on the image of TQFTs in the target$d$- category (more precisely, the sub-groupoid of fully dualizable objects and invertible morphisms).

The best-known case of oriented surfaces, when $G = \mathrm{SO}(2)$, requires a *Calabi-Yau* structure on $Z(+)$. This can be variously phrased: as a trivialization of the *Serre functor*, which is an automorphism of any fully dualizable linear dg category (see Remark 2.3 below); alternatively, as a linear functional on the cyclic homology of $Z(+)$ whose restriction to Hochschild homology $HH_*(Z(+))$ induces a perfect pairing on Hom spaces:

$$\mathrm{Hom}(x,y)\otimes\mathrm{Hom}(y,x)\to\mathrm{Hom}(x,x)\to HH_*\to\mathbb{C}.$$

This case of Lurie's theorem recovers earlier results of Costello, Kontsevich and Hopkins-Lurie [C, KS].

The Hochschild homology $HH_*(Z(+))$ is meaningful in a different guise: it is the space of states $Z(S^1)$ of the theory, for the circle with the radial framing. The circle is pictured here with a germ of surrounding surface, and therefore carries a $\mathbb{Z}$'s worth of framings, detected by a winding number. The Hochschild cohomology HH^* goes with the blackboard framing, and the space for the framing with winding number n is HH^* of the nth power of the Serre functor. (Of course, for oriented theories there is no framing dependence, and these spaces agree.)

2.2. Topological group actions. An important point is that the action of $\mathrm{O}(2)$ (and thus G) on the target category $Z(+)$ is *topological*, or factored through its topology. There are several ways to formulate this constraint, which is vacuous when G is discrete. The favored formulation will depend on the nature of the target category; in the linear case, and when G is connected, we will provisionally settle for the one in Theorem 2.5 below. Combined with Statement 2.9 below, this generalizes an old result of Seidel [Sei] on Hamiltonian diffeomorphism groups.

Here are some alternative definitions:

1. We can ask for a *local trivialization* of the action in a contractible neighborhood of $1\in G$, an isomorphism with the trivial action of that same neighborhood (up to coherent homotopies of all orders).

2. Using the action to form a bundle of categories with fiber $Z(+)$ over the classifying stack BG, we ask for an integrable flat connection on the resulting bundle of categories. (Formulating the flatness condition requires some care, in light of the fiber-wise automorphisms.)

3. Exploiting the contractibility of the group P_1G of paths starting at $1\in G$, we can ask for a trivialization of the lifted P_1G-action.

   Now, the action of the based loop group ΩG (kernel of $P_1G\to G$) is already trivial (being factored through $1\in G$), and the difference of trivializations defines a (topological) representation of ΩG by automorphisms of the identity functor in $Z(+)$.

   The group ΩG has an E_2 structure, seen from its equivalence with the second loop space Ω^2BG; and the representation on $\mathrm{Id}_{Z(+)}$ is the 2-holonomy, over spheres, of the flat connection in #2. Importantly, it is an E_2 representation.

Remark 2.2. When G is connected, description #3 above captures all the information for the action (up to contractible choices), because the space of trivializations of a trivial topological action of P_1G is contractible.

Example 2.3. A topological action of the circle on a category is given by a group homomorphism from $\mathbb{Z} = \pi_1 S^1 = \pi_0 \Omega S^1$ to the automorphisms of the identity: equivalently, a central (in the category) automorphism of each object. Because there is no higher topology in S^1, this also works when the target is a 2-category, such as the (sub-groupoid of fully dualizable objects in the) 2-category $\mathfrak{LCat}$, the structural $\mathrm{SO}(2) \subset \mathrm{O}(2)$ action gives an automorphism of each category: this is the Serre functor.

Example 2.4. Endomorphisms of the identity in the linear category $\mathfrak{Vect}$ are the complex scalars, so that linear topological representations of a connected G on $\mathfrak{Vect}$ are 1-dimensional representations of $\pi_0 \Omega G \cong \pi_1 G$. These are the points in the center of the complexified Langlands dual group $G^\vee_{\mathbb{C}}$.

Recall that the endomorphisms of the identity in a category (the center) form the 0^{th} Hochschild cohomology. To generalize the above example to the derived world, we should include the entire Hochschild cochain complex.

Theorem 2.5. *Topological actions of a connected group G on a linear dg-category $\mathfrak{C}$ are captured (up to contractible choices) by the induced E_2 algebra homomorphism from the chains $C_*\Omega G$, with Pontrjagin product, to the Hochschild cochains of $\mathfrak{C}$.* □

Example 2.6. From a continuous action of G on a space X, we get a locally trivial action on the cochains C^*X. Indeed, we get an action of ΩG on the free loop space LX of X. The action is fiber-wise with respect to the bundle $\Omega X \to LX \to X$. Let $C^*\big(X; C_*\tilde{\Omega}X\big)$ be the cochain complex on X with coefficients in the fiber-wise chains for this bundle. With the fiber-wise Pontrjagin product, this is a model for the Hochschild cochains of the algebra $C^*(X)$, and the action of ΩG exhibits the E_2 homomorphism in the theorem.

Remark 2.7. The "E_2" in the statement is not jus a commutativity constraint, but can contain (infinite amounts of!) data; see Lesson 3.2.5.

Remark 2.8. One floor below, for 1-dimensional field theories, the category $Z(+)$ is replaced with a vector space (or a complex), and we recognize #2 above as defining a topological representation of G. The datum in Theorem 2.5 is replaced by an (E_1) algebra homomorphism from the chains C_*G, with Pontrjagin product, to $\mathrm{End}(Z(+))$; there is no connectivity assumption. Climbing to the higher ground of n-categories, we can extract an E_{n+1}-algebra homomorphism from $C_*\Omega^n G$ to the E_n Hochschild cohomology; but this misses the information from the homotopy of G below n.

The following key example captures the relevance of my story to real mathematics. (In fact, it contains *all* examples I know for topological group actions!)

Conjecture 2.9. *Let G act in Hamiltonian fashion action on a symplectic manifold X. Then, G acts topologically on the Fukaya category of X.*

Proof. A Hamiltonian action of G on X defines, in the category of symplectic manifolds and Lagrangian correspondences, an action of the group object T^*G.[1] This makes the Fukaya category of X into a module category over the wrapped Fukaya category $\mathfrak{WF}(T^*G)$. A theorem of Abouzaid [Ab] identifies the latter with that of $C_*\Omega G$-modules. The tensor structure is identified with the E_2 structure of the Pontrjagin product, by detecting it on generators of the category (the cotangent fibers). The resulting structure is equivalent to the datum in Theorem 2.5. □

Remark 2.10. It may seem strange to state a conjecture and then provide a proof. However, the reader will detect certain assumptions which have not been clearly stated in the conjecture: mainly, functoriality of Fukaya categories under Lagrangian correspondences. If X is non-compact, equivariance of the wrapping condition at ∞ is essential; the statement fails for the *infinitesimally wrapped* Fukaya category of Nadler and Zaslow [NZ], see below. (Another outline argument is more tightly connected to holomorphic disks and $G_{\mathbb{C}}$-bundles, but that relies on details of the construction of the Fukaya category.)

Remark 2.11. A closely related notion to the one discussed, but distinct from it, is that of an *infinitesimally trivialized* Lie group action. Here, we ask for the action to be differentiable, and the restricted action to the formal group $\hat{G}$ (equivalently, the Lie algebra $\mathfrak{g}$) should be homologically trivialized. An example is furnished by an action of G on a manifold X and the induced action on the algebra $\mathcal{D}(X)$ of differential operators: the Lie action of $\mathfrak{g}$ is trivialized in the sense that it is inner, realized by the natural Lie homomorphism from $\mathfrak{g}$ to the 1st order differential operators. Theorem 2.5 does *not* usually apply to such situations. With respect to the alternative definition #2 above, the relevant distinction is between *flat* and *integrable* connections over BG.

2.3. Gauging a topological theory.

Given a guantum field theory and a (compact Lie) group G, physicists normally produce a G-gauged theory in two stages. The theory is first coupled to a 'classical gauge background', a principal G-bundle. (No connection is needed in the case of topological actions.[2]) Then, we 'integrate over all principal bundles' to quantize the gauge theory.

These two distinct stages are neatly spelt out in the setting of extended TQFTs. Lurie's theory already captures the first stage of gauging. Namely, we convert the principal G-bundle into a tangential structure by choosing the trivial homomorphism $G \to \mathrm{O}(2)$. (Of course, we may add any desired tangential structure, such as orientability, by switching to $G \times \mathrm{SO}(2) \to \mathrm{O}(2)$, by projection.) Making $Z(+)$ into a fixed point for the trivial G-action means defining a (topological) G-action on $Z(+)$. This is the input datum for a classically gauged theory.

Quantizing the gauge theory, or integrating over principal G-bundles, is tricky. It is straightforward for finite groups: integration of numbers is a weighted sum,

[1] The moment map $\mu : X \to \mathfrak{g}^*$ appears in the requisite Lagrangian, $\{(g, \mu(gx), x, gx)\} \subset T^*G \times (-X) \times X$.

[2] Flat connections would be needed when G action does not factor through topology, as in B-model theories.

and integration of vector spaces and categories is a finite limit or colimit. (The duality constraints require the limits and colimts to agree; working in characteristic 0 ensures that [FHLT].) For Lie groups G, integration of the numbers requires a fundamental class on the moduli of principal bundles. For instance, the symplectic volume form is relevant to topological Yang-mills theory. A limited K-theoretic fundamental class was defined in [TW], and cohomological classes, such as the one relevant to topological Yang-Mills theory, can be extracted from it. But this matter seems worthy of more subtle discussion than space allows here.

In fact, the gauge theory *cannot* always be fully quantized. The generating object for the quantum gauge theory is the invariant category $Z(+)^G$, which agrees with the co-invariant category $Z(+)_G$ under mild assumptions. In the framework of Theorem 2.5, we compute the generator $Z(+)_G$ as a tensor product

$$Z(+)_G = Z(+) \otimes_{C_*\Omega G} \mathfrak{Vect} \tag{2.1}$$

with the trivial representation. The 1-dimensional part of the field theory, and sometimes part of the surface operations, are well-defined; but the complete surface-level operations often fail to be defined. Thus, for the trivial $2D$ theory, $Z(+) =$ dg$-\mathfrak{Vect}$ with trivial G-action, and the fixed-points are local systems over BG. This generates a partially defined $2D$ theory, a version of string topology for the space BG. The space associated to the circle is the equivariant cohomology $H^*_G(G)$ for the conjugation action, and the theory is defined the subcategory of $Bord_2$ where all surfaces (top morphisms) have non-empty output boundaries for each component.

This example can be made more interesting by noting that the trivial action of G on dg$-\mathfrak{Vect}$ has interesting topological deformations, in the $\mathbb{Z}/2$-graded world; the notable one comes from the quadratic Casimir in $H^4(BG)$, and gives topological Yang-Mills theory with gauge group G. When G is semi-simple, this theory is almost completely defined, and the invariants of a closed surface (of genus 2 or more) are the symplectic volumes of the moduli spaces of flat connections. (Further deformations exits, by the entire even cohomology of BG and relate to more general integrals over those spaces.) These should be regarded as twisted Gromov-Witten theories with target space BG. A starting point of the present work was the abject failure of the homological calculation (2.1) in these examples: for topological Yang-Mills theory, (2.1) gives the zero answer when G is simple.

2.4. The space of states. Independently of good behavior of the fixed-point category $Z(+)^G$, the space(s) of states of the gauged theory are well-defined. More precisely, each $g \in G$ gives an autofunctor g_* of the category. The Hochshild cochain complexes $HCH^*(g_*; Z(+))$ assemble to a (derived) local system $\mathcal{H}(Z(+))$ over the group G, which is equivariant for the conjugation action, and the space of states for the (blackboard framed) circle in the gauge theory is the equivariant homology $H^G_*(G; \mathcal{H})$. It has a natural E_2 multiplication, using the Pontrjagin product in the group. When $Z(+) = \mathfrak{Vect}$, with the trivial G-action, we recover the string topology space $H^G_*(G)$ of BG by exploiting Poincaré duality on G.[3]

[3]The last space goes with the radially framed circle.

3. The 2-category of Kapustin-Rozansky-Saulina

As the image of the point, an object in the 3-dimensional bordism 3-category, Lurie's generator for pure 3-dimensional gauge theory should have categorical depth 2. My proposal for this generator is a 2-category associated to a certain holomorphic symplectic manifold, to be described in §5.

Fortunately, the existence of the requisite 2-category has already been conjectured, and a proposal for its construction has been outlined in [KRS, KR]. When X is compact, this 2-category should generate the Rozansky-Witten theory [RW] of X. In particular, its Hochschild cohomology, which on general grounds is a 1-category with a braided tensor structure, should be (a dg refinement of) the derived category of coherent sheaves on X described in [RW2]. Just like Rozansky-Witten theory, the narrative takes place in a differential graded world, and in applications, the integer grading must be collapsed $\mod 2$ (the symplectic form needs to have degree 2, if the integral grading is to be kept). To keep the language simple, I will use 'sheaf' for 'complex of sheaves' and write $\mathfrak{Coh}$ for a differential graded version of the category of coherent sheaves, etc.

Remark 3.1. The 2-category may at first appear analogous to the deformation quantization of the symplectic manifold; but that is not so. That analogue — a double categorification — is $\mathfrak{Coh}(X)$ with its braided tensor structure. The category [KRS] is a 'square root' of that, and I will denote it $\sqrt{\mathfrak{Coh}}(X)$ or $KRS(X)$.

3.1. Simplified description.

The following partial description of the KRS 2-category applies to a Stein manifold X, when deformations coming from coherent cohomology vanish.[4] In our example, X will be affine algebraic. Among objects of $\sqrt{\mathfrak{Coh}}(X)$ are smooth holomorphic Lagrangians $L \subset X$; more general objects are coherent sheaves of $\mathcal{O}_L$-linear categories on such L. (The object L itself stands for its dg category $\mathfrak{Coh}(L)$ of coherent sheaves, a generator for the above.) To make this even more precisee, $\sqrt{\mathfrak{Coh}}(X)$ is the sheaf of global sections of a coherent sheaf of $\mathcal{O}_X$-linear 2-categories, whose localization at any smooth L as above is equivalent the 2-category of module categories over the sheaf of tensor categories $(\mathfrak{Coh}(L), \otimes)$ on L; with a bit of faith, this pins down $\sqrt{\mathfrak{Coh}}(X)$, as follows.

For two Lagrangians $L, L' \in X$, $\mathrm{Hom}(L, L')$ will be a sheaf of categories supported on $L \cap L'$, and a $(\mathfrak{Coh}(L), \otimes) - (\mathfrak{Coh}(L'), \otimes)$ bi-module. Localizing at L, we choose a (formal) neighborhood identified symplectically with T^*L, so that we regard (locally) L' as the graph of a differential $d\Psi$, for a *potential* function $\Psi : L \to \mathbb{C}$. Locally where this identification is valid, $\mathrm{Hom}(L, L')$ becomes equivalent to the *matrix factorization* category $MF(L, \Psi)$. (See for instance [O].)

3.2. Lessons.

Several insights emerge from this important notion.

1. A familiar actor in mirror symmetry, a complex manifold L with potential Ψ, is really the object in $\sqrt{\mathfrak{Coh}}(T^*L)$ represented by the graph $\Gamma(d\Psi)$, masquerading as a more traditional geometric object. The matrix factorization

[4] My discussion is faulty in another way, failing to incorporate the Spin structures, which must be carried by the Lagrangians. I am grateful to D. Joyce for flagging their role.

category $MF(L,\Psi)$ is its Hom with the zero-section. This resolves the contradiction in which the restriction of the category $MF(L,\Psi)$ to a sub-manifold $M \subset L$ is commonly taken to be the matrix factorization category of $\Psi|_M$. That is clearly false in the 2-category of $(\mathfrak{Coh}(L),\otimes)$-module categories (the result of localizing to the zero-section $L \subset T^*L$). For instance, if the critical locus of Ψ does not meet M, Hom computed in $(\mathfrak{Coh}(L),\otimes)$-modules gives zero. Instead, M must be replaced by the object represented by its co-normal bundle in $\sqrt{\mathfrak{Coh}}(T^*L)$, whose Hom there with $\Gamma(d\Psi)$ computes precisely $MF(M,\Psi|_M)$.

2. The well-defined assignment sends $(\mathfrak{Coh}(L),\otimes)$-module categories to sheaves of categories with Lagrangian support in the cotangent bundle $\hat{T}^*L$, completed at the zero-section. Namely, the Hochschild cohomology of such a category $\mathfrak{K}$ is (locally on L) an E_2-algebra over the second (E_2) Hochschild cohomology of $(\mathfrak{Coh}(L),\otimes)$, which is an E_3 algebra. The spectrum of the latter is $\hat{T}^*L$, with E_3 structure given by the standard symplectic form. This turns $\operatorname{Spec} HH^*(\mathfrak{K})$ into a coherent sheaf with co-isotropic support in $\hat{T}^*L$, and $\mathfrak{K}$ sheafifies over it. The *Lagrangian* condition is clearly related to a finiteness constraint, but this certainly shows the need to include singular Lagrangians in the KRS 2-category.

3. The deformation of a $(\mathfrak{Coh}(L),\otimes)$-module category $\mathfrak{M}$ by the addition of a potential ('curving') $\Psi \in \mathcal{O}(L)$ shifts the support of $\mathfrak{M}$ vertically by $d\Psi$ in T^*L. This allows one to move from formal to analytic neighborhoods of L, if the deformation theory under curvings is well-understood. For instance, one can compute the Hom between two objects that do not intersect the zero-section — such as two potentials without critical points — by drawing their intersection into L: $\mathrm{Hom}(\Gamma(d\Phi),\Gamma(d\Psi)) = MF(L,\Psi-\Phi)$.

4. More generally, Hamiltonian vector fields on $\hat{T}^*L$ give the derivations of $\sqrt{\mathfrak{Coh}}(\hat{T}^*L)$ defined from its E_2 Hochschild cohomology. Hamiltonians vanishing on the zero-section preserve the latter, and give first-order automorphisms of $(\mathfrak{Coh}(L),\otimes)$.

5. The KRS picture captures in geometric terms sophisticated algebraic information. For example, the category $\mathfrak{Vect}$ can be given a $(\mathfrak{Coh}(L),\otimes)$-module structure in many more ways in the $\mathbb{Z}/2$ graded world: any potential Ψ with a single, Morse critical point will accomplish that. The location of the critical point $p \in L$ misses an infinite amount of information, which is captured precisely by the graph of $d\Psi$; this is equivalent to an E_2 structure on the evaluation homomorphism $\mathcal{O}_L \to \mathbb{C}_p$ at the residue field (cf. Theorem 2.5).

Parts of this story can be made rigorous at the level of formal deformation theory, see for instance [F], and of course the outline in [KR]. Lesson 3 also offers a working definition of the 2-category $\sqrt{\mathfrak{Coh}}(T^*L)$ as that of $(\mathfrak{Coh}(L),\otimes)$-modules, together with all their deformations by curvings. On a general symplectic manifold X, we can hope to patch the local definitions from here.[5] It is not my purpose

[5] If X is not Stein, deformations will be imposed upon this story by coherent cohomology.

to supply a construction of $\sqrt{\mathfrak{Coh}}(X)$ here — indeed, that is an important open question — but rather, to indicate enough structure to explain my answer to the mirror of (non-abelian) gauge theory. I believe that one important reason why that particular question has been troublesome is that the mirror holomorphic symplectic manifold, the BFM space of §5, *not* quite a cotangent bundle, so the usual description in terms of complex manifolds with potentials is inadequate.

Remark 3.2. If $X = T^*L$ for a manifold L, and we insist on integer, rather than $\mathbb{Z}/2$-gradings, then the cotangent fibers have degree 2 and all structure in the KRS category is invariant under the scaling action on T^*L. In that case, we are dealing precisely with $(\mathfrak{Coh}(L), \otimes)$-modules.

3.3. Boundary conditions and domain walls. The Hom category $\mathrm{Hom}(L, L')$ for two Lagrangians $L, L' \subset X$ with finite intersection supplies a 2-dimensional topological field theory for framed surfaces; this follows form its local description by matrix factorizations. Since X itself aims to define a $3D$ (Rozansky-Witten) theory and each of L, L' is a boundary condition for it, one should picture a sandwich of Rozansky-Witten filling between a bottom slice of L and a top one of L'. The formal description is that $L, L' : \mathrm{Id} \to RW_X$ are morphisms from the trivial $3D$ theory Id to Rozansky-Witten theory RW_X, viewed as functors from $Bord_2$ to the 3-category of linear 2-categories, and the category $\mathrm{Hom}(L, L')$ of natural transformations between these morphisms is the generator for this sandwich theory. Geometrically, it is represented by the interval, with RW_X in the bulk and L, L' at the ends, and is also known as the *compactification* of RW_X along the interval, with the named boundary conditions.

Factoring this theory through *oriented* surfaces requires a trace on the Hochschild homology HH_* (cf. §2.1). Now, the canonical description of the only non-zero group, $HH_{\dim L}$, turns out to involve the Spin square roots[6] of the canonical bundles ω, ω' of L, L' on their scheme-theoretic overlap:

$$HH_{\dim L}\,\mathrm{Hom}(L, L') \cong \Gamma\big(L \cap L'; (\omega \otimes \omega')^{1/2}\big). \tag{3.1}$$

A non-degenerate quadratic form on $HH_{\dim L}$ comes from the Grothendieck residue (and the symplectic volume on X). A non-degenerate trace on HH_* will thus be defined by choosing non-vanishing sections of $\omega^{1/2}, \omega'^{1/2}$ on L, L'.

Remark 3.3. A generalization of the notion of boundary condition is that of a *domain wall* between TQFTs. This is an adjoint pair of functors between the TQFTs meeting certain (dualizability) conditions, see [L], §4. A boundary condition is a domain wall with the trivial TQFT. Just as a holomorphic Lagrangian in X can be expected to define a boundary condition for RW_X, a holomorphic Lagrangian correspondence $X \leftarrow C \rightarrow Y$ should define a domain wall between RW_X and RW_Y. We shalll use these in §5 and §6, in comparing gauge theories for different groups.

[6]The *cohomology* is easy to pin down canonically, as the functions on $L \cap L'$.

4. The mirror of abelian gauge theory

This interlude recalls the mirror story of torus gauge theory; except for the difficulty mentioned in Lesson 1 of §3.2, this story is well understood and can be phrased as a categorified Fourier-Mukai transform. In fact, in this case we can indicate the other mirror transformation, from the gauged B-model to a family of A-models.

4.1. The $\mathbb{Z}$-graded story.

We will need to correct this when abandoning $\mathbb{Z}$-gradings, in light of the wisdom of the previous section; nevertheless the following picture is nearly right.

Proposition 4.1. *(i) Topological actions of the torus T on the category $\mathfrak{Vect}$ are classified by points in the complexified dual torus $T_{\mathbb{C}}^{\vee}$.*
(ii) A topological action of T on a linear category $\mathfrak{C}$ is equivalent to a quasi-coherent sheafification of $\mathfrak{C}$ over $T_{\mathbb{C}}^{\vee}$.

Proof. Both statements follow from Theorem 2.5, considering that the group ring $C_*(\Omega T)$ is quasi-isomorphic to the ring of algebraic functions on $T_{\mathbb{C}}^{\vee}$, and that a category naturally sheafifies over its center, the zeroth Hochschild cohomology. □

There emerges the following 0^{th} order approximation to abelian gauged mirror symmetry: if X is a symplectic manifold with Hamiltonian action of T, and $X^{\vee}$ is a mirror of X — in the sense that $\mathfrak{Coh}(X^{\vee})$ is equivalent to the Fukaya category $\mathfrak{F}(X)$ — then the group action on X is mirrored into a holomorphic map $\pi : X^{\vee} \to T_{\mathbb{C}}^{\vee}$. This picture could be readily extracted from Seidel's result, [Sei].

Proposition 4.1 interprets the mirror map $X^{\vee} \to T_{\mathbb{C}}^{\vee}$ as a *spectral decomposition* of the category $\mathfrak{F}(X)$ into irreducibles $\mathfrak{Vect}_{\tau}$. One of the motivating conjectures of this program gives a geometric interpretation of this spectral decomposition, in terms of the original manifold X and the moment map $\mu : X \to \mathfrak{t}^*$.

Conjecture 4.2 (Torus symplectic quotients). *The multiplicity of $\mathfrak{Vect}_{\tau}$ in $\mathfrak{F}(X)$ is the Fukaya category of the symplectic reduction of X at the point $\operatorname{Re}\log\tau \in \mathfrak{t}^*$, with imaginary curving (B-field) $\operatorname{Im}\log\tau$.*

Remark 4.3. This is, for now, meaningless over singular values of the moment map, where there seems to be no candidate definition for the Fukaya category of the quotient.

Remark 4.4. The conjecture relies on using the *unitary mirror* of X, constructed from Lagrangians with unitary local systems. Otherwise, in the toric case, the algebraic mirror $X^{\vee}$ is $T_{\mathbb{C}}^{\vee}$, obviously having a point fiber over every point in $T_{\mathbb{C}}^{\vee}$; yet the symplectic reduction is empty for values outside the moment polytope. That polytope is precisely the cut-off prescribed for the mirror by unitarity.

Example 4.5 (Toric varieties). The following construction of mirrors for toric manifolds, going back to the work of Givental and Hori-Vafa, illustrates both the conjecture and the need to correct the picture by moving to the KRS category.

Start with the mirror of $X = \mathbb{C}^N$, with standard symplectic form, as the space $T_{\mathbb{C}}^{\vee} := (\mathbb{C}^{\times})^N$ with potential $\Psi = z_1 + \ldots z_N$. Here, $T^{\vee}$ is the dual of the diagonal torus acting on X, and the mirror map $X^\vee \to T^\vee_\mathbb{C}$ is the identity.[7] For a sub-torus $i : K \hookrightarrow T$, the mirror of the symplectic reduction $X_q := \mathbb{C}^N/\!/_q K$ at $q \in \mathfrak{k}^*$ is the (torus) fiber $X^\vee_q$ of the dual surjection $i^\vee : T^\vee_\mathbb{C} \twoheadrightarrow K^\vee_\mathbb{C}$, with restricted superpotential Ψ. The parameter q lives in the small quantum cohomology of X. We see here the familiar, but faulty restriction to the fiber of the matrix factorization category $MF(T^\vee_\mathbb{C}, \Psi)$ of Lesson 3.2, #1. The problem is glaring, because the original MF category is null.

The mirror $X^\vee_q$ projects isomorphically to the kernel $S^\vee_\mathbb{C}$ of $i^\vee$; this is the map π mirror to the action of $S = T/K$ on X.

4.2. Fourier transform. As can be expected in the abelian case, the spectral decomposition of Proposition 4.1 is formally given by a Fourier transform. Specifically, there is a 'categorical Poincaré line bundle'

$$\mathfrak{P} \to BT_\mathbb{C} \times T^\vee_\mathbb{C},$$

with an integrable flat connection along BT. (Of course, $\mathfrak{P}$ is the universal one-dimensional topological representation of T, and its fiber over $\tau \in T^\vee_\mathbb{C}$ is $\mathfrak{Vect}_\tau$.) Given a category $\mathfrak{C}$ with topological T-action, we form the bundle $\mathrm{Hom}(\mathfrak{P}, \mathfrak{C})$ and integrate along $BT_\mathbb{C}$ to obtain the spectral decomposition of $\mathfrak{C}$ laid out over $T^\vee_\mathbb{C}$.

Remark 4.6 (B to A). The interest in this observation stems from a related Fourier transformation, giving a "B to A" mirror symmetry. There is another Poincaré bundle $\mathfrak{Q} \to BT_\mathbb{C} \times T^\vee_\mathbb{C}$, with flat structure this time along $T^\vee_\mathbb{C}$. It may help to exploit flatness and descend to $B(T_\mathbb{C} \times \pi_1(T)^\vee)$, in which case $\mathfrak{Q}$ is the line $\mathfrak{Vect}$ with action of the group $T \times \pi_1(T)^\vee$, defined by the Heisenberg $\mathbb{C}^\times$-central extension. (The extension is a multiplicative assignment of a line to every group element, and the action on $\mathfrak{Vect}$ tensors by that line.)

Fourier transform converts a category $\mathfrak{C}$ with (non-topological!) T-action into a local system $\tilde{\mathfrak{C}}$ of categories over $T^\vee_\mathbb{C}$. The fiber of $\tilde{\mathfrak{C}}$ over 1 is the fixed-point category $\mathfrak{C}^T$, and the monodromy action of $\pi_1(T^\vee)$ comes from the natural action thereon of the category $\mathfrak{Rep}(T)$ of complex T-representations. For example, when $\mathfrak{C} = \mathfrak{Coh}(X)$, the (dg) category of coherent sheaves on a complex manifold with holomorphic T-action, $\mathfrak{C}^T$ is, almost by definition, the category of sheaves on the quotient stack $X/T_\mathbb{C}$. The analogue of Conjecture 4.2 is completely obvious here.

I do not know a non-abelian analogue of this "B to A" story.

4.3. The $\mathbb{Z}/2$-graded story. In light of Lesson 3.2.1 and Example 4.5, the only change needed to reach the true story is to replace the $(\mathfrak{Coh}(T^\vee_\mathbb{C}), \otimes)$-module category $\mathfrak{Coh}(X^\vee)$, determined from $\pi : X^\vee \to T^\vee_\mathbb{C}$, by an object in the KRS category of $T^*T^\vee_\mathbb{C}$: the category with T-action sees precisely the germ of a KRS object near the zero-section.

This enhancement of information relies upon knowing not just the Fukaya category $\mathfrak{F}(X)$ with its torus action, but all of its curvings with respect to functions

[7] This is readily obtained from the SYZ picture, using coordinate tori as Lagrangians; the unitary mirror is cut off by $|z_k| < 1$.

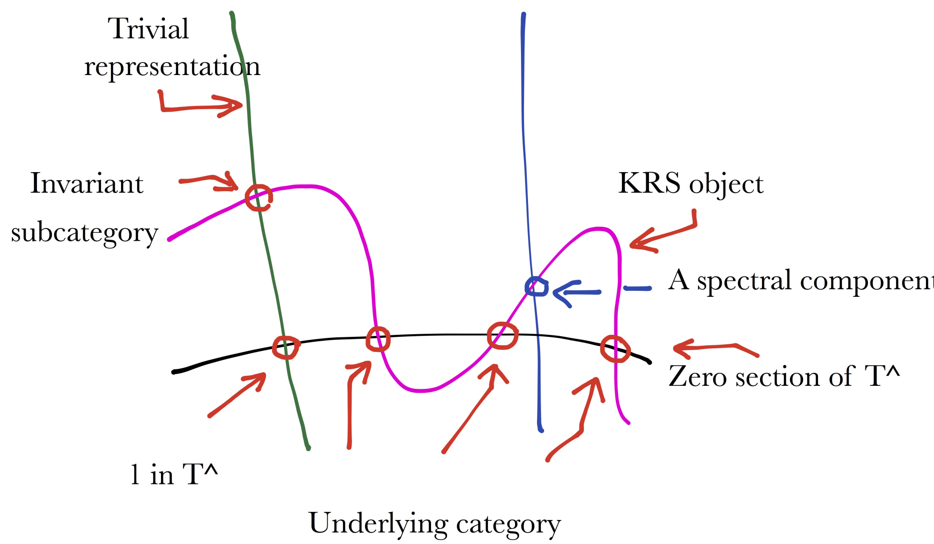

Figure 1. Pictorial representation of $\sqrt{\mathfrak{Coh}}(T^*T_{\mathbb{C}}^{\vee})$

lifted from the mirror map $\pi : X^{\vee} \to T_{\mathbb{C}}^{\vee}$. However, we can expect in examples that a meaningful geometric construction of the mirror would carry that information. For instance, in Example 4.5, we replace $(X_q^{\vee}, \Psi|_{X_q^{\vee}})$ and its map to $S_{\mathbb{C}}^{\vee}$ by the graph of $d\Psi|_{X_q^{\vee}}$ in $T^*S_{\mathbb{C}}^{\vee}$; this is the result of intersecting the graph of $d\Psi$ with the cotangent space at $q \in K_{\mathbb{C}}^{\vee}$.

Figure 1 attempts to capture the distinction between $(\mathfrak{Coh}T_{\mathbb{C}}^{\vee}, \otimes)$-modules and their KRS enhancement. The squiggly line stands for (the support of) a general object; its germ at the zero-section is the underlying category, with topological T-action. In that sense, the zero-section represents the regular representation of T (its Hom category with any object recovers the underlying category.) The invariant category is the intercept with the trivial representation, the cotangent space at $1 \in T_{\mathbb{C}}^{\vee}$; other spectral components are intercepts with vertical axes. We see that the invariant subcategory is computed 'far' from the underlying category, and a homological calculation centered at the zero-section will fail.

5. The non-abelian mirror $BFM(G^{\vee})$

For torus actions, the insight was that gauging a Fukaya category $\mathfrak{F}(X)$ amounted to enriching it from a $\mathfrak{Coh}(T_{\mathbb{C}}^{\vee})$ module to an object in $\sqrt{\mathfrak{Coh}}(T^*T_{\mathbb{C}}^{\vee})$. In a cotangent bundle, this promotion may seem modest. A non-abelian Lie group G will move us to a more sophisticated holomorphic algebraic manifold which is *not* a cotangent bundle. Let T be a maximal torus of G, W the Weyl group and B, B_+ two opposite (lower and upper triangular) Borel subgroups, N, N_+ their unipotent radicals; Fraktur letters will stand for the Lie algebras and $^\vee$ will indicate their counterparts in the Langlands dual Lie group $G^\vee$.

5.1. The home of $2D$ gauge theory. The space $BFM(G)$ was introduced and studied by Bezrukavnikov, Mirkovic and Finkelberg [BFM] in general, but special instances were known in many guises. Here are several descriptions. Call $T^*_{\mathrm{reg}}G_{\mathbb{C}} \subset T^*G_{\mathbb{C}}$ the Zariski-open subset comprising the *regular* cotangent vectors (centralizer of minimal dimension, the rank of G).

Theorem 5.1. *The following describe the same holomorphic symplectic manifold, denoted $BFM(G)$.*
(i) The spectrum of the complex equivariant homology $H_*^{G^\vee}(\Omega G^\vee)$, with Pontrjagin multiplication.
(ii) The holomorphic symplectic reduction of $T^*_{\mathrm{reg}}G_{\mathbb{C}}$ by conjugation under $G_{\mathbb{C}}$.
(iii) The affine resolution of singularities of the quotient $T^*T_{\mathbb{C}}/W$, obtained by adjoining the functions $(e^\alpha - 1)/\alpha$. (α ranges over the roots of $\mathfrak{g}$, $e^\alpha - 1$ is the respective function on $T_{\mathbb{C}}$ and the denominator α is the linear function on $\mathfrak{t}^*$.)
(iv) $BFM(\mathrm{SU}_n)$ is the moduli space of SU_2 monopoles of charge n, and is a Zariski-open subset of the Hilbert scheme of n points in $T^*\mathbb{C}^\times$ [AH].
(v) $BFM(T) = T^*T_{\mathbb{C}}$

Remark 5.2. The moment map zero-fiber for the conjugation $G_{\mathbb{C}}$-action on $T^*_{\mathrm{reg}}G_{\mathbb{C}}$ is the (regular) *universal centralizer* $\mathscr{Z}_{\mathrm{reg}} = \{(g,\xi) \mid g\xi g^{-1} = \xi, \xi \text{ is regular}\}$. $\mathscr{Z}_{\mathrm{reg}}$ is smooth, and $BFM(G) = \mathscr{Z}_{\mathrm{reg}}/G_{\mathbb{C}}$, with stabilizer of constant dimension and local slices. This is the only one of the descriptions that makes the holomorphic symplectic structure evident.

The space $BFM(G^\vee)$ inherits two projections from $T^*_{\mathrm{reg}}G_{\mathbb{C}}$: π_v, to the space $(\mathfrak{g}^\vee)^*_{\mathbb{C}}/G^\vee_{\mathbb{C}} \cong \mathfrak{t}_{\mathbb{C}}/W$ of co-adjoint orbits, and π_h, to the conjugacy classes in $G^\vee_{\mathbb{C}}$. Both are Poisson-integrable with Lagrangian fibers. The projection π_v will have the more obvious meaning for gauge theory, capturing the $H^*(BG)$-module structure on fixed-point categories. The projection π_h is closely related to the restriction to T (and to the *string topology* of flag varieties.)

The symplectic structure on $BFM(G^\vee)$ relates to its nature as (an uncompletion of) the second Hochschild cohomology of the E_2-algebra $H_*(\Omega G)$.[8] In fact, $BFM(G)$ contains the zero-fiber of π_v, $Z := \mathrm{Spec}H_*(\Omega G)$, as a smooth Lagrangian; it comes from the part of $\mathscr{Z}_{\mathrm{reg}}$ with nilpotent ξ (cf. Remark 5.2).

Theorem 2.5 and Lesson 3.2.2 sheafify categories with topological G-action over the formal neighborhood of Z. However, it is the entire space $BFM(G^\vee)$ which is the correct receptacle for G-gauge theory: gauged TQFTs are objects in the 2-category $\sqrt{\mathfrak{Coh}}(BFM(G^\vee))$. Clearly, that requires a rethinking of the notion: the definition of 'topological category with G-action' as in §2 would complete the BFM space at the exceptional Lagrangian Z. Loosely speaking, we need to know a theory together with all its deformations of the group action.

[8]Of course, the E_2 structure is trivial over the complex numbers and the algebra is quasi-isomorphic to its underlying dg ring of chains.

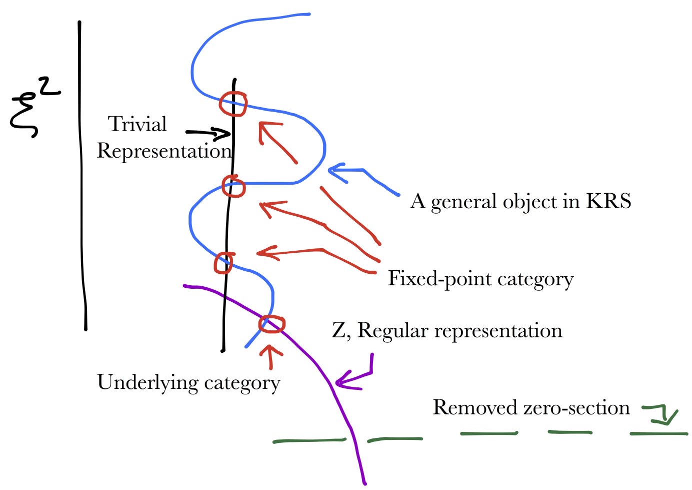

Figure 2. BFM space of PSU_2; the fiber of π_h at 1 is $Z\cup$trivial representation

The Lagrangian Z replaces the zero-section from the torus case, and plays the role of the regular representation of G: $\mathrm{Hom}(Z, L)$ gives the underlying category of the representation L. The formal calculation is $\mathrm{Hom}_{C_*\Omega G}(C_*\Omega G; L) = L$, if we use Theorem 2.5 to model representations. Figure 2 below sketches $BFM(\mathrm{PSU}_2)$.

5.2. Induction by String topology. No map relates $BFM(T^\vee) = T^*T^\vee_{\mathbb{C}}$ and $BFM(G^\vee)$, because of the blow-up, but a holomorphic Lagrangian correspondence is defined from the branched cover

$$BFM(G^\vee) \longleftarrow BFM(G^\vee) \times_{\mathfrak{t}_{\mathbb{C}}/W} \mathfrak{t}_{\mathbb{C}} \longrightarrow T^*T^\vee_{\mathbb{C}}. \tag{5.1}$$

The right map is neither proper not open.[9] A holomorphic Lagrangian correspondences could give a pair of adjoint functors between the respective $\sqrt{\mathfrak{Coh}}$ 2-categories, thus a domain wall between T- and G- gauge theories (cf. §3.3). This is indeed the case, and we can identify the functors.

Theorem 5.3. *The correspondence* (5.1) *matches an adjoint pair of restriction-induction functors between categorical T- and G-representations. Induction from a category $\mathfrak{C}$ with topological T-action is effected by* string topology with coefficients *of the flag variety G/T:*

$$\mathrm{Ind}(\mathfrak{C}) = C_*\Omega G \otimes_{C_*\Omega T} \mathfrak{C}.$$

Restriction is the obvious functor. □

[9] Z maps to $1 \in T^\vee$, but most of the zero-section in $T^*T^\vee_{\mathbb{C}}$ is missed by the map.

Remark 5.4. (i) An alternative (slightly worse) description of induction is given by the category of (derived) global sections $\mathrm{R}\Gamma\big(G/T;\tilde{\mathfrak{C}}\big)$ for the associated local system $\tilde{\mathfrak{C}}$ of categories.
(ii) Neither description is quite correct. Just as the BFM spaces carry more information than the category and the action, so does induction.
(iii) For example, inducing from the representation $\mathfrak{Vect}_\tau$, for a point $\tau \in T^\vee_{\mathbb{C}}$ which is *not* central in G, by either method above, will appear to give zero. (This is what a homological algebra calculation of the curved string topology of G/T for a non-trivial curving $\tau \in H^2(G/T;\mathbb{C}^\times)$ gives.) However, geometric induction gives the fiber of $\pi_v^{-1}(\tau)$. The puzzle is resolved by noting that none of those fibers meet the regular representation Z, so the underlying categories are null. We are letting G act on categories without objects, and growing wiser.
(iv) The 'naïvely induced' representations can serve to probe the entire BFM space by abelianization. It is therefor not *conceptually* more difficult to understand non-abelian gauged mirrors than abelian ones. However, the symplectically induced representations of the next section are much nicer.

5.3. Alternative model for induction. I close with a new model for the correspondence (5.1), useful in a later mirror calculation. Call $\mathfrak{b}_{+,\mathrm{reg}} \subset \mathfrak{b}_+$ the open subset of regular elements. Identify $\mathfrak{b}_+ = \big(\mathfrak{g}_{\mathbb{C}}/\mathfrak{n}_+\big)^*$, B_+-equivariantly; the last space matches the fibers of the bundle, over $B_+ \subset G_{\mathbb{C}}$, of co-normals to the N_+-translation orbits. Using this to define the left map below and projection on the right gives a holomorphic Lagrangian correspondence

$$\begin{tikzcd}[column sep=small] & \dfrac{B_+ \times \mathfrak{b}_{+,\mathrm{reg}}}{B_+} \arrow[dl] \arrow[dr] & \\ T^*_{\mathrm{reg}} G_{\mathbb{C}}/\!/_{\mathrm{ad}} B_+ & & T^*T_{\mathbb{C}} \end{tikzcd}$$

having divided by the conjugation action of $B_+^\vee$. We can also divide out by B_+ in the defining correspondence for $BFM(G)$,

$$BFM(G) \longleftarrow \mathcal{Z}_{\mathrm{reg}}/B_+ \longrightarrow T^*_{\mathrm{reg}} G_{\mathbb{C}}/\!/_{\mathrm{ad}} B_+.$$

The composition of these two can be shown to yield (5.1) (for the group G).

6. Mirrors of flag varieties

I will now explain the place of flag varieties in the mirror view of gauge theory. Lifting to the torus-equivariant picture will recover a construction of K. Rietsch [R].

6.1. Flag varieties as domain walls. Let $L \subset G$ be a Levi subgroup, centralizer of a dominant weight $\lambda : \mathfrak{l} \to \mathrm{i}\mathbb{R}$. The flag variety $X = G/L$ is a symplectic manifold with Hamiltonian G-action (the co-adjoint orbit of λ), and as such it should have a mirror holomorphic Lagrangian in $BFM(G^\vee)$. This will be true, but we forgot some structure relevant to gauge theory. Namely, we can use G/L to *symplectically induce* categorical representations from L to G.

A categorical representation $\mathfrak{C}$ of L gives the local system of categories $\tilde{\mathfrak{C}} = G \times_L \mathfrak{C} \to X$, and we can construct the Fukaya category of X with coefficients in $\tilde{\mathfrak{C}}$. (Objects would be horizontal sections of objects over Lagrangians, and Floer complexes can be formed in the usual way from the Hom-spaces over intersections.) In fact, the weight λ (or rather, its exponential e^λ in the center of $L^\vee_{\mathbb{C}}$) defines a topological representation $\mathfrak{Vect}_\lambda$ of L, and we can think of the ordinary Fukaya category $\mathfrak{F}(X,\lambda)$ as the symplectic induction from the latter. The precise meaning is that deforming λ in $\mathfrak{Vect}_\lambda$ achieves the same effect as the matching deformation of the symplectic form. An imaginary variation of λ (movement in the unitary group $L^\vee$) has the effect of adding a unitary B-field twist to the Fukaya category.

Remark 6.1. Left adjoint to the symplectic induction functor SInd_L^G is a *symplectic restriction* from G to L. This is not the ordinary (forgetful) restriction, which instead is adjoint to string topology induction (§5). For example, when $L = T$, the spectral decomposition under T of the symplectic restriction of $\mathfrak{C}$ would extract the multiplicities of the $\mathfrak{F}(X,\tau)$ in $\mathfrak{C}$, rather than those of the $\mathfrak{Vect}_\tau$.

This pair of functors is a new *domain wall* between pure 3-dimensional G- and L-gauge theories. On the mirror side, we can hope to represent a domain wall by a holomorphic Lagrangian correspondence between $BFM(L^\vee)$ and $BFM(G^\vee)$. We will be fortunate to identify this correspondence as an open embedding.

To recover the mirror of X in its various incarnations (as a symplectic manifold, or a G-equivariant symplectic one) we must apply boundary conditions to the two gauge theories, aiming for the 'sandwich picture' of a $2D$ TQFT, as in §3.3. For example, to find the underlying symplectic manifold (X,λ), we must apply the representation $\mathfrak{Vect}_\lambda$ of L and the regular representation Z of G. I shall carry out this (and a more general) exercise in the final section.

The study of symplectically induced representations can be motivated by the following conjecture, the evident non-abelian counterpart of Conjecture 4.2 (with the difference that it seems much less approachable).

Conjecture 6.2. *For a Hamiltonian G-action on the compact symplectic manifold X and a regular value μ of the moment map, the Fukaya category $\mathfrak{F}(X/\!/G)$, reduced at the orbit of μ (and with unitary B-field $\mathrm{i}\nu$) is the multiplicity in $\mathfrak{X}$ of the representation symplectically induced from $\mathfrak{Vect}_{\mu+i\nu}$.*

6.2. The Toda isomorphism. The following isomorphism of holomorphic symplectic manifolds is mirror to symplectic induction. It fits within a broad range of related results ('Whittaker constructions') due to Kostant. Its relation to Fukaya categories of flag varieties is mysterious, and now only understood with reference to the appearance of the Toda integrable system in the Gromov-Witten theory of flag varieties [GK, K]. From that point of view, the isomorphism enhances the Toda system by supplying the conjugate family of commuting Hamiltonians, pulled back from conjugacy classes in the group, rather than orbits the Lie algebra.

The mirror picture of G-gauge theory involves the Langlands dual group $G^\vee$ of G, but the notation is cleaner with G. With notation as in §5, call $\chi : \mathfrak{n} \to \mathbb{C}^\times$ the regular character (unique up to $T_\mathbb{C}$-conjugation) and consider the *Toda space*, the holomorphic symplectic quotient of $T^*G_\mathbb{C}$

$$T(G) := (N, \chi)\backslash\!\backslash T^*G_\mathbb{C}/\!/(N, \chi)$$

under the left×right action of N, reduced at the point $(\chi, \chi) \in \mathfrak{n}^* \oplus \mathfrak{n}^*$.

Theorem 6.3. *We have a holomorphic symplectic isomorphism*

$$T(G) = (N, \chi)\backslash\!\backslash T^*G_\mathbb{C}/\!/(N, \chi) \cong T^*_{\mathrm{reg}}G_\mathbb{C}/\!/_{\mathrm{Ad}}G_\mathbb{C} = BFM(G)$$

induced from the presentation of the two manifolds as holomorphic symplectic reductions of the same manifold $T^*_{\mathrm{reg}}G_\mathbb{C}$.

Proof. The $N \times N$ moment fiber in $T^*G_\mathbb{C} \cong G_\mathbb{C} \times \mathfrak{g}^*_\mathbb{C}$ (by left trivialization) is

$$\mathcal{T} := \{(g, \xi) \in G_\mathbb{C} \times \mathfrak{g}^*_\mathbb{C} \mid \pi(\xi) = \pi(g\xi g^{-1}) = \chi\},$$

where $\pi : \mathfrak{g}^*_\mathbb{C} \to \mathfrak{n}^*$ is the projection. As $\pi^{-1}(\chi)$ consists of regular elements, we may use $T^*_{\mathrm{reg}}G_\mathbb{C}$ instead. Now, N acts freely on $\pi^{-1}(\chi)$, with Kostant's global slice, so the $N \times N$ action on $\mathcal{T}$ is free also and $T(G) = N\backslash\mathcal{T}/N$ is a manifold.

The moment map fibers $\mathcal{T}$ and $\mathcal{Z}_{\mathrm{reg}}$ (for the Ad-action of $G_\mathbb{C}$) provide holomorphic Lagrangian correspondences

$$\begin{tikzcd}[column sep=small] & \mathcal{T} \arrow[dl] \arrow[dr] & & \mathcal{Z}_{\mathrm{reg}} \arrow[dl] \arrow[dr] & \\ T(G) & & T^*_{\mathrm{reg}}G_\mathbb{C} & & BFM(G) \end{tikzcd} \tag{6.1}$$

whose composition $\mathcal{T} \times_{T^*_{\mathrm{reg}}G_\mathbb{C}} \mathcal{Z}_{\mathrm{reg}}$, I claim, induces an isomorphism. Actually, the clean correspondence must mind the fact that the two actions on T^*G, of $N \times N$ and G, respectively, have in common the conjugation action of N (sitting diagonally in $N \times N$): so we must really factor through $T^*_{\mathrm{reg}}G_\mathbb{C}/\!/_{\mathrm{Ad}}(N)$, within which the co-isotropics $\mathcal{T}/_{\mathrm{Ad}}N$ and $\mathcal{Z}_{\mathrm{reg}}/_{\mathrm{Ad}}N$ turn out to intersect transversally.

We check that the composition in (6.1) induces a bijection on points: preservation of the Poisson structure then supplies the Jacobian criterion. Choose $(g, \xi) \in \mathcal{T}$; then, $\xi, g\xi g^{-1} \in \pi^{-1}(\chi)$ are in the same $G_\mathbb{C}$-orbit in $\mathfrak{g}^*_\mathbb{C}$. Kostant's slice theorem ensures that the two elements are then Ad-related by a unique $\nu \in N$, $\nu g\xi(\nu g)^{-1} = \xi$. There is then, up to right action of N, a unique $(g', \xi') \in \mathcal{Z}_{\mathrm{reg}}$ in the $N \times N$-orbit of (g, ξ). We thus get an injection $T(G) \hookrightarrow BFM(G)$. To see surjectivity, conjugate a chosen $(h, \eta) \in \mathcal{Z}_{\mathrm{reg}}$ to bring η into $\pi^{-1}(\chi)$. The result is in $\mathcal{T}$ (and is again unique up to N-conjugation). $\square$

Remark 6.4. The space $T(G)$ has a hyperkähler structure; it comes from a third description, as a moduli space of solutions to Nahm's equations. This is closely related to a conjectural derivation of my mirror conjecture (6.5) below from Langlands (electric-magnetic) duality in 4-dimensional $N = 4$ Yang-Mills theory. (I am indebted to E. Witten for this explanation.)

6.3. The mirror of symplectic induction. Inclusion of the open cell $N \times w_0 \cdot T_{\mathbb{C}} \times N \subset G_{\mathbb{C}}$ leads to a holomorphic symplectic embedding $T^*T_{\mathbb{C}} \subset T(G)$. Sending a co-tangent vector to its co-adjoint orbit projects $T(G)$ to $\mathfrak{g}^*_{\mathbb{C}}//G^{ad}_{\mathbb{C}}$, and the functions on the latter space lift to the commuting Hamiltonians of the Toda integrable system; so the theorem completes the picture by providing a complementary set of Hamiltonians lifted from the conjugacy classes of G.

More generally, if $L \subset G$ is a Levi subgroup, with representative $w_L \in L$ of its longest Weyl element, and with unipotent group $N_L = N \cap L_{\mathbb{C}}$, then χ restrict to a regular character of N_L and the inclusion

$$N \times_{N_L} w_0 w_L^{-1} \cdot L_{\mathbb{C}} \times_{N_L} N \subset G_{\mathbb{C}}$$

determines an open embedding $T(L) \subset T(G)$. The following is, among others, a character formula for induced representations. It relies on too many wobbly definitions to be called a theorem, but assuming it is meaningful, its truth can be established form existing knowledge.

Conjecture 6.5. *Via the Toda isomorphism, the embedding $T(L^\vee) \subset T(G^\vee)$ is mirror to symplectic induction from L to G, representing the flag variety G/L as a domain wall between L- and G-gauge theories.*

Example 6.6. With the torus $L = T$, a one-dimensional representation of T is described by a point $q \in T^\vee$, represented in $\sqrt{\mathfrak{Coh}}(T^*T^\vee)$ by the cotangent space at q. Its image under the Toda isomorphism, a Lagrangian leaf $\Lambda(q) \subset BFM(G)$, is the symplectically induced representation, or the G-equivariant Fukaya category of the flag variety G/T with quantum parameter q. The analogue of the character is the structure sheaf $\mathcal{O}_{\Lambda(q)}$, whose algebra of global sections is the G-equivariant quantum cohomology of G/T [GK].

Remark 6.7. It is difficult to prove the conjecture without a precise definitions (of equivariant Fukaya categories with coefficients and of the KRS 2-category). Nevertheless, accepting that $BFM(G^\vee)$ as the correct mirror of G-gauge theory, the conjecture follows from known results about the equivariant quantum cohomology of flag varieties [GK, C-F, Mi]. The latter describe $qH^*_G(G/L)$ as a module over $H^*(BG) = \mathbb{C}[\mathfrak{g}]^G$, the algebra of Toda Hamiltonians, induced from the projection π_v. The symplectic condition turns out to pin the map uniquely.

6.4. Foliation by induced representations. Recall (Example 2.4) the one-dimensional representations of a Levi subgroup $L \subset G$, corresponding to the points in the center of $L^\vee_{\mathbb{C}}$. Let us call them *cuspidal*: they are not symplectically induced from a smaller Levi subgroup. (Such a symplectic induction produces representation of rank equal to the Euler characteristic of the flag variety.) The following proposition suggests that these induced representations are better suited to spectral theory that the naïvely induced ones of §5.

Theorem 6.8. *The space $BFM(G^\vee)$ is smoothly foliated by symplectic inductions of cuspidal representations: each leaf comes from a unique cuspidal representation of a unique Levi subgroup L, with $T \subset L \subset G$.*

Proof. The leaves are the fibers of $N\backslash\mathcal{T}^\vee/N \to N\backslash G^\vee_{\mathbb{C}}/N$, and induction on the semi-simple rank reduces us to checking that the part of $\mathcal{T}^\vee$ which does *not* come from any $T(L^\vee)$, for a proper $L \subset G$, lives over the center of $G^\vee_{\mathbb{C}}$.

Omit $^\vee$ from the notation and choose $(g,\xi) \in \mathcal{T}$. From $G_{\mathbb{C}} = \coprod_w N \cdot wT_{\mathbb{C}} \cdot N$, we may take $g \in wT_{\mathbb{C}}$ for some $w \in W$. Split $\mathfrak{g}^*_{\mathbb{C}} = \mathfrak{n}^* \oplus \mathfrak{t}^*_{\mathbb{C}} \oplus \mathfrak{n}^*_+$; then,

$$\begin{aligned} \xi &= \chi + \eta + \nu, && \text{for some } \eta \in \mathfrak{t}_{\mathbb{C}}, \nu \in \mathfrak{n}^*_+ \\ g\xi g^{-1} &= \chi + w(\eta) + \nu', && \text{for some } \nu' \in \mathfrak{n}^*_+ \end{aligned}$$

whence we see that w sends each simple negative root either to a simple negative root, or to a positive root. If $w = 1$, then $g \in T_{\mathbb{C}}$ centralizes $\chi \pmod{\mathfrak{b}^*_+}$ and thus lies in the center of $G_{\mathbb{C}}$. Otherwise, I claim that $w = w_0 w_L^{-1}$, for the Levi L whose negative simple roots stay negative. Equivalently, the unique simple root system of $\mathfrak{g}$ comprising the simple negative roots of L and otherwise only positive roots, is the w_L-transform of the positive root system. This can be seen by choosing a point $\zeta + \varepsilon$, with ζ generic on the L-fixed face of the dominant Weyl chamber, and ε a dominant regular displacement: $w_L(\zeta + \varepsilon)$ must be in the dominant chamber of the new root system. □

Example 6.9 ($G = \mathrm{SU}_2$). The dual complex group is $G^\vee_{\mathbb{C}} = \mathrm{PSL}_2(\mathbb{C})$, whose BFM space is the blow-up of $\mathbb{C} \times \mathbb{C}^\times/\{\pm 1\}$ at $(0,1)$, with the proper transform of the zero-section $\{0\} \times \mathbb{C}^\times/\{\pm 1\}$ removed. This is the Atiyah-Hitchin manifold studied in [AH]. The $\mathbb{Z}/2$-action identifies (ξ, z) with $(-\xi, z^{-1})$. Projection to the line of co-adjoint orbits is given by the Toda Hamiltonian ξ^2.

The Toda inclusion of $T^*T^\vee_{\mathbb{C}} \cong \mathbb{C} \times \mathbb{C}^\times$ sends a point (u,q) to

$$\xi^2 = u^2 - q, \qquad \frac{z + z^{-1}}{4} = \frac{u^2}{q} - \frac{1}{2}$$

(A match of signs is required between z and ξ.) The induced leaves of constant q are given by

$$\xi = q\frac{\sqrt{z} - \sqrt{z}^{-1}}{2},$$

after lifting to the coordinates $\xi, \sqrt{z}$ for the double-cover maximal torus in SL_2. We recognize here the (graph of the differentiated potential in the) S^1-equivariant mirror of the flag variety $\mathbb{P}^1$.

The one remaining leaf in $BFM(\mathrm{PSU}_2)$ is the trivial representation of SU_2; it is the proper transform of $T^*_1\mathbb{C}^\times/\{\pm 1\}$, the image in $\mathbb{C} \times \mathbb{C}^\times/\{\pm 1\}$ of) the cotangent fiber at 1. If we switch instead to $\mathrm{PSU}(2)$, the new BFM space (on the Langlands dual side) is a double cover of the former, and there is a new cuspidal leaf over the central point $(-\mathrm{I}_2) \in \mathrm{SU}_2$, corresponding to the sign representation of $\pi_1\mathrm{PSU}_2$.

6.5. Torus-equivariant flag varieties. Restricting the G-action to T, the flag manifold G/L is a transformation from L-gauge theory to T-gauge theory, given by composition of the symplectic induction and string topology domain walls:

$$T(L^\vee) \xhookrightarrow{\mathrm{SInd}} T(G^\vee) \xrightarrow[\mathrm{Toda}]{\sim} BFM(G) \xrightarrow{\mathrm{ST}} BFM(T^\vee) = T^*T^\vee_{\mathbb{C}} \tag{6.2}$$

The equivariant mirror is a family of $2D$ TQFTs, which can be defined, for instance, by a family of complex manifolds with potentials parametrized by the Lie algebra $\mathfrak{t}_{\mathbb{C}}$. This family reflects the $H^*(BT)$-module structure on equivariant quantum cohomology. When $\mathfrak{F}(G/L)$ has been represented by an object $\Lambda \in \sqrt{\mathfrak{Coh}}(T^*T_{\mathbb{C}}^{\vee})$, the family comes from the projection of $T^*T_{\mathbb{C}}^{\vee}$ to the cotangent fiber, and the TQFTs are the fibers of Λ over $\mathfrak{t}_{\mathbb{C}}$, the Hom categories with the constant sections of $T^*T_{\mathbb{C}}^{\vee}$.

To recover this family of mirrors from the double domain wall (6.2), we must use it to pair two Lagrangians, in $T(L^{\vee})$ and in $T^*T_{\mathbb{C}}^{\vee}$. The Lagrangians are

- the Lagrangian leaf $\Lambda(q) \subset BFM(L^{\vee})$ over a point q in the center of $L_{\mathbb{C}}^{\vee}$, describing a cuspidal representation of L (q is also the quantum parameter for G/L);
- the constant Lagrangian section S_{ξ} of $T^*T_{\mathbb{C}}^{\vee}$, with fixed value $\xi \in \mathfrak{t}_{\mathbb{C}}$.

Note that S_{ξ} is the differential of a multi-valued character $\xi \circ \log : T_{\mathbb{C}}^{\vee} \to \mathbb{C}$.

Remark 6.10. The relevant TQFT picture is a sandwich with triple-decker filling: the base slice is the representation $\mathfrak{Vect}_q$ of L corresponding to $\Lambda(q)$, a boundary condition for L-gauge theory. The filling of the sandwich is a triple layer of L, G, T gauge theories, separated by the SInd and string topology domain walls in (6.2). The sandwich is topped with the slice S_{ξ}, a boundary condition for T-gauge theory. Its underlying representation category is null, if $\xi \neq 0$; S_{ξ} is a deformation of the regular representation of T by the multi-valued potential $\xi \circ \log$.

6.6. Rietsch mirrors.

Building on ideas of Peterson and earlier calculations of Givental-Kim, Ciocan-Fontanine, Kostant and Mihalcea [GK, C-F, K, Mi], Rietsch [R] proposed torus-equivariant complex mirrors for all flag varieties G/L.

Let us recover these from my story by computing the answer outlined above. Recall (§5.3) the Lagrangian correspondence

$$T^*T_{\mathbb{C}} \leftarrow B_+ \times \mathfrak{b}_{+,\mathrm{reg}} \to T^*_{\mathrm{reg}}G_{\mathbb{C}},$$

appearing in the alternate model for the string topology induction. Compose this with the Toda construction to define the following holomorphic Lagrangian correspondence between $T(G)$ and $BFM(T) = T^*T_{\mathbb{C}}$:

$$\begin{tikzcd} & \mathcal{T} \arrow[dl, "P"'] \arrow[dr] & & B_+ \times \mathfrak{b}_{+,\mathrm{reg}} \arrow[dl] \arrow[dr, "p"] & \\ T(G) & & T^*_{\mathrm{reg}}G_{\mathbb{C}} & & T^*T_{\mathbb{C}} \end{tikzcd} \tag{6.3}$$

Proposition 6.11. *Correspondence* (6.3) *is the composition* ST $\circ$ Toda *of* (6.2).

Sketch of proof. In the jagged triangle of correspondences below, the left edge is the Toda isomorphism, the right edge the correspondence (6.3) and the bottom edge the string topology domain wall. The long, counterclockwise way from top to right involves division by the complementary subgroups N and B_+ of $G_{\mathbb{C}}$; so it seems reasonable that the composition should agree with the undivided correspondence (6.3) on the right edge:

$$\begin{tikzcd}
& \mathcal{T}/_{\mathrm{Ad}}(N) \arrow[rr] \arrow[d] & & \boxed{\mathrm{T(G)}} & & \mathcal{T} \arrow[ll] \arrow[d] & \mathcal{T} \cap \big(B_+ \times \mathfrak{b}_{+,\mathrm{reg}}\big) \arrow[l, dotted] \arrow[d, dotted] \\
\mathcal{Z}_{\mathrm{reg}}/_{\mathrm{Ad}}(N) \arrow[r, tail] \arrow[ddd] & T^*_{\mathrm{reg}}G_{\mathbb{C}}/\!/_{\mathrm{Ad}}N & & \mathcal{Z}_{\mathrm{reg}}(B_+)/B_+ \arrow[dl, dotted] \arrow[dr, dotted] & & T^*_{\mathrm{reg}}G_{\mathbb{C}} & B_+ \times \mathfrak{b}_{+,\mathrm{reg}} \arrow[l] \arrow[ddd] \\
& & \dfrac{\mathcal{Z}_{\mathrm{reg}}}{B_+} \arrow[dll] \arrow[dr] & & \dfrac{B_+ \times \mathfrak{b}_{+,\mathrm{reg}}}{B_+} \arrow[dl] \arrow[drr] & & \\
\boxed{\mathrm{BFM(G)}} & & & T^*_{\mathrm{reg}}G_{\mathbb{C}}/\!/_{\mathrm{Ad}}B_+ & & & \boxed{\mathrm{BFM(T)}}
\end{tikzcd}$$

The argument exploits the regularity of the Lie algebra elements. The intersection in the upper right corner comprises the pairs $(b, \beta) \in B_+ \times \mathfrak{b}_+$ with b centralizing $\beta \in E + \mathfrak{t}_{\mathbb{C}}$. ($E = \chi$ under $\mathfrak{n}_+ \cong \mathfrak{n}^*$.) That is a slice for the conjugation B_+-action on the regular centralizer $\mathcal{Z}_{\mathrm{reg}}(B_+)$ in B_+, which makes clear the isomorphism with the fiber product in the center the triangle; and the map is compatible with the Toda isomorphism on the left edge. $\square$

We now calculate the pairing $S_\xi \subset T(L^\vee)$ and $\Lambda(q) \subset T^*T^\vee_{\mathbb{C}}$ by the correspondence (6.3) for the dual group $G^\vee$. We do so by computing in $T^*_{\mathrm{reg}}G^\vee_{\mathbb{C}}$

$$\mathrm{Hom}\left(p^{-1}S_\xi, P^{-1}\Lambda(q)\right).$$

The two Lagrangians meet over the intersection

$$B_+^\vee \cap \left(N^\vee \cdot w_0 w_L^{-1} L^\vee_{\mathbb{C}} \cdot N^\vee\right) \subset G^\vee_{\mathbb{C}}.$$

Lift $\xi \circ \log$ to $B_+^\vee$ by p; over $B_+^\vee$, $p^{-1}S_\xi$ is the conormal bundle to $B_+^\vee \subset G^\vee_{\mathbb{C}}$ shifted by the graph of $d(\xi \circ \log)$. (The shifted bundle is well-defined, independently of any local extension of the function $\xi \circ \log$.)

The Lagrangian $P^{-1}\Lambda(q)$ lives over the open set $N^\vee \cdot w_0 w_L^{-1} L^\vee_{\mathbb{C}} \cdot N^\vee$ in $G^\vee_{\mathbb{C}}$, where it is the shifted co-normal bundle to the submanifold

$$M := N^\vee \cdot w_0 w_L^{-1} q \cdot N^\vee \cong \frac{N^\vee \times N^\vee}{\mathrm{diag}(N^\vee \cap L^\vee_{\mathbb{C}})},$$

shifted into $\mathcal{T}$ by the graph of the differential of the following function f:

$$f : n_1 \cdot w_0 w_L^{-1} l \cdot n_2 \mapsto \chi(\log n_1 + \log n_2).$$

Now, $B_+^\vee$ and M meet transversally in $G_\mathbb{C}^\vee$, in a manifold isomorphic to a Zariski-open in the flag variety $G^\vee/L^\vee$; this is the $\mathcal{R}_{w_0,w_L}$ of [R]-. Transversality permits us to dispense with the conormal bundles, and identify $\mathrm{Hom}(S_\xi, \Lambda(q))$ with the pairing, in the cotangent bundles, between graphs of the restricted functions to $B_+^\vee \cap M$

$$\mathrm{Hom}_{T^*(B_+^\vee \cap M)}\left(\Gamma(d(\xi \circ \log)), \Gamma(df)\right);$$

this is the matrix factorization category $MF\big(B_+^\vee \cap M; f - \xi \circ \log\big)$. This is the Rietsch mirror of G/L.

The last mirror comes with a volume form, which defines the trace on HH_*. In the Lagrangian correspondence, we need instead a half-volume form on each leaf. The two leaves S_ξ and $\Lambda(q)$ do in fact carry natural half-volumes, translation-invariant for the groups (B and $N \times N$) and along the cotangent fibers. Rietsch's volume form on the mirror $\mathcal{R}_{w_0,w_L}$ comes from the product of these half-volumes.

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