Gauge Theory, Mirror Symmetry, and Langlands Duality

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Gromov-Witten theory

Among topological quantum field theories studied in past decades, Gromov-Witten theory in 2D has enjoyed enduring interest.

This associates a compact symplectic manifold X a *space of states* $H^*(X)$. *Correlators* assigned to surfaces with points labeled by states count the pseudo-holomorphic maps to X with incidence conditions.

When the surfaces vary in a family, the numerical invariants refine to cohomology classes on the Deligne-Mumford spaces $\overline{M}_g^n$.

These invariants could contain enormous information, and a structural classification has not been accomplished in general. (Exception, Givental's conjecture in semi-simple case: toric varieties, Grassmannians,...)

Mirror symmetry (Lerche, Vafa, Warner and refined by many others) promises to reduce GW theory to more standard computations in the complex geometry of a conjectural mirror manifold $X^\vee$.

It has been checked in many examples (toric Fanos, hypersurfaces therein).

Homological mirror symmetry

was introduced by Kontsevich to spell out the structure of the invariants.

His key idea: extend the "closed string" theory to an "open-closed" one, involving surfaces with corners, and "boundary conditions" forming a linear category with structure (Frobenius, or Calabi-Yau, or cyclic A_∞). This category, with enough structure, should determine all invariants.

On the symplectic side (X, ω), this is Fukaya's A_∞ category $\mathcal{F}(X)$; on the complex side, $D^b Coh(X^\vee)$ (with its Yoneda structure); plus ?.

Homological Mirror symmetry, the conjectural match of the two structured categories, has been verified in many examples: $K3$ (Seidel); del Pezzo surfaces, weighted projective spaces (Auroux, Katzarkov, Orlov); toric Fanos (FO^3+ Abouzaid+others); Calabi-Yau hypersufaces (Sheridan)

The mirror of a toric variety X with torus $T_{\mathbb{C}}$ is the dual torus $T_{\mathbb{C}}^\vee$, with standard volume, plus a *super-potential* Ψ, a Laurent polynomial . The associated category of Ψ-*Matrix factorizations* is $\mathbb{Z}/2$-graded only.

Group actions and Hamiltonian quotients

Many GW computations involve *Hamiltonian quotients* of simpler varieties. Thus, projective toric varieties are quotients of vector spaces by linear torus actions. Their mirrors can be described in those terms.

Example (Givental-Hori-Vafa mirror; simplified)

The best-known case is $\mathbb{P}^{n-1} = \mathbb{C}^n /\!/ \mathrm{U}(1)$, with mirror

$$(\mathbb{C}^*)^{n-1} = \{(z_1, \ldots, z_n) | z_1 z_2 \cdots z_n = q\}, \Psi = z_1 + \cdots + z_n$$

For $Y = \mathbb{C}^n$, with standard $(\mathbb{C}^*)^n$ action, declare the mirror to be

$$Y^\vee = (\mathbb{C}^*)^n, \quad \Psi = z_1 + \cdots + z_n.$$

For $X = \mathbb{C}^n /\!/ K$ with $K_\mathbb{C} \subset (\mathbb{C}^*)^n$, $X^\vee_\mathbf{q}$ is the fiber over $\mathbf{q} \in K^\vee_\mathbb{C}$ of the dual surjection $(\mathbb{C}^*)^n \twoheadrightarrow K^\vee_\mathbb{C}$, and the super-potential is the restricted Ψ. (The *Novikov variables* $\mathbf{q}$ track degrees of holomorphic curves.)

Mirror of a Lie group action. Langlands dual group

Addressing HMS in relation to Hamiltonian quotients raises the following

Basic Questions

1. Find the mirror structure on $X^\vee$ for a Hamiltonian group action on X.
2. In terms of this structure, what is the mirror to the GIT quotient?

Basic Answers (Torus case; 0th order approximation)

1. The mirror to a T-action on (X, ω) is a holomorphic map $X^\vee \to T^\vee_{\mathbb{C}}$.
2. The mirror of $X//T$ is the fiber of $X^\vee$ over 1.

Basic Answers (Compact, connected G; (-1)st order approximation)

1. The mirror to a G-action on X is a holomorphic map from $X^\vee$ to the space of conjugacy classes in the Langlands dual group $G^\vee_{\mathbb{C}}$.
2. Not worth stating yet (the fiber over 1 is wrong).

Problem with the answers: they are wrong

Pursuing them in the context of HMS quickly leads to paradoxes.

Thus, in the GHV mirror, the original Ψ has no critical points on Y, so its matrix factorization category is zero.

We will not get the (interesting) Fukaya category of a toric variety by gauging the zero category; and indeed, GHV tell us to *first* restrict Ψ to the fiber of $(\mathbb{C}^*)^n \to K_{\mathbb{C}}^{\vee}$ *and then* compute MF.

On the face of it, this operation is not defined in terms of categories.

The right answers can be found using arguments from 4D QFT; the tour covers some beautiful geometry. (I learnt these ideas from Ed Witten.)

This beautiful story is a fairy-tale for two reasons:

1. it is not rigorous, 2. history did not happen this way.

SU(2) magnetic monopoles

are solutions of the **Bogomolny equation** $\boxed{F = *D\phi}$ on $\mathbb{R}^3$, where F is the curvature of an $\mathfrak{su}(2)$-connection D and ϕ is valued in the ad-bundle.

They correspond to time-invariant ASD connections $D + \phi\, dt$ on $\mathbb{R}^4$.
Finiteness of the energy breaks the symmetry to U(1) on the sphere at ∞, leading to a discrete invariant, the monopole charge $n \geq 0$.

Monopole moduli spaces were studied by Atiyah, Donaldson, Hitchin, Manton, Nahm, ... and shown to be hyper-Kähler manifolds.
The charge n moduli space was described in several ways; among them,

1. A specific Zariski-open subset of the nth Hilbert scheme of $\mathbb{C}^* \times \mathbb{C}$; this is a partial resolution of singularities of $T^*(\mathbb{C}^*)^n/S_n$, and is naturally associated to the group U(n).
2. The space of solutions of *Nahm's equations* $\frac{dT_i}{ds} + \varepsilon_i^{jk} T_j T_k = 0$, $T_i(s) \in \mathfrak{su}(n)$, simple poles with $\mathrm{Res}_{s=0,2} T_i$ giving the irrep of SU(2).

3D reduction of Yang-Mills theory

Seiberg and Witten studied the reduction of $4D$ Yang-Mills theory (which has a topological version giving the Donaldson invariants) along a circle.

In the 0 radius limit, they described the low-energy regime of the SU(2) theory as a Sigma-model in the space of vacua, which they identified as the Atiyah-Hitchin monopole moduli space of charge 2.

Thus, they identified the 3-dimensional SU(2) gauge theory, got from 4D by reducing along a circle of zero radius, with the Rozansky-Witten theory of the hyper-Kähler Atiyah-Hitchin manifold.

Subsequent work (Argyres-Farragi) described SU(n) gauge theory, in terms of the charge n monopole space. Martinec-Warner discussed a general G in terms of the so-called *periodic Toda integrable system*, revealing a first connection to the Langlands dual Lie algebra $\mathfrak{g}^\vee$.

(They did not quite give a complete description of the space.)

An equivalence of field theories identifies their (2-)categories of branes.

Now, boundary conditions for the $3D$ pure gauge theory are general 2D gauged TQFTs: categories with (locally trivial) G-action.

Branes for RW theory were recently described by Kapustin and Rozansky.
They form a 2-category, generated by smooth holomorphic Lagrangians L.
Locally, the category $End(L)$ is the tensor category $D^b Coh(L)$.
A L' near L can be written as the graph of $d\Psi$; $Hom(L, L')$ is the category of matrix factorizations of Ψ over L. It's supported on $L \cap L'$.
The global description is deformed by the ambient symplectic manifold.

In other words, localized branes at L are $\mathcal{O}$-linear categories on L.
Kapustin and Rozansky assert that these Lagrangian neighborhoods can be patched to a (sheaf of) 2-categories.

Example (Cotangent bundle T^*L)

The matrix factorizations for a $\Psi \in \mathcal{O}(L)$ constitute $Hom(L; \Gamma(d\Psi))$.
This micro-localization of the MF category as a brane circumvents a number of difficulties in the theory of curved algebras and categories.

Key features of an A-model group action

Theorem

1. *A Hamiltonian action of G on (X, ω) induces a locally trivial action of G on $\mathcal{F}(X)$. (Trivialized near $1 \in G$)*
2. *This is described (up to homotopy) by a morphism of E_2-algebras*
$$C_*(\Omega G) \to HCH^*(\mathcal{F}(X));$$
or, a module category structure of $\mathcal{F}(X)$ over $(C_*(\Omega G)$-modules, $\otimes)$.
3. *The invariant part $\mathcal{F}(X)^G$ is the fiber over $0 \in Spec\, C_*(\Omega G)$ of $\mathcal{F}(X)$.*
4. *The latter "gauged category" should be closely related to $\mathcal{F}(X/\!/G)$.*

Remark

$H_*(\Omega G; \mathbb{C})$ is a (Laurent) polynomial ring, and is truly commutative (E_∞).
The same is expected of $HH^*(\mathcal{F}(X))$ (at least, when $\cong H^*(X)$).
But an E_2 morphism between commutative algebras contains *more information* than the underlying morphism of algebras.

The monopole and Rozansky-Witten connection

Theorem (Bezrkukavnikov-Finkelberg-Mirkovic)

1. *Spec* $H_*^G(\Omega G)$ *is an affine resolution of singularities of* $(T^* T_{\mathbb{C}}^\vee)/W$.
2. *Spec* $H_*(\Omega G)$ *is fiber in Spec* $H_*^G(\Omega G)$ *over* $Z(G^\vee) \subset (T_{\mathbb{C}}^\vee)/W$.
3. *Spec* $H_*^G(\Omega G)$ *is algebraic symplectic, and Spec* $H_*(\Omega G)$ *Lagrangian. Completed there,* $H^G(\Omega G)$ *is the* E_2 *Hochschild cohomology of* $H_*(\Omega G)$ *(a.k.a. the cotangent bundle.)*

Remark

1. This $E_2 HH^*$ controls the formal E_2 deformations of $H_*(\Omega G)$.
   Algebras with E_2-action of $H_*(\Omega G)$ micro-localize to Spec$(E_2 HH^*)$, defining branes for the Rozansky-Witten theory.
   The BFM space provides a natural *uncompletion* for this.
2. $G = \mathrm{SU}(n)$ gives the SU(2)−monopole space of charge n.
   General: solns. to *Nahm's equations* in $\mathfrak{g}^\vee$ with principal $\mathfrak{sl}(2)$ poles.

Theorem (Sort of; BFM description of gauge theory)

The BFM un-completion governs A-models gauged by G.
Specifically, a G-action on a Fukaya category gives the germ of a brane in the RW theory of Spec $H_*^G(\Omega G)$, near the Lagrangian $H_*(\Omega G)$.
Gauging the theory requires extending this to a brane in the ambient space.

Remark

1. This theorem is partially a definition. Specifically, we are giving a notion of a locally trivial G-action on a linear category, precise enough to specify the gauged theory (the fixed-point category).
2. This un-completion strictifies a homotopy G-action to a "genuine" G-action, and is analogous to passing from $K(BG)$ to $K_G(*)$.
3. The 2-category of linear categories with locally trivial G-action has a forgetful "underlying category" functor.
   Unlike the case of G-action on vector spaces, this is not faithful.
   So the description in terms of a (locally trivial, up to coherent homotopy) group action on a category was only a starting point.

Underlying category and invariant category

In the RW model, we must still describe geometrically two functors from (2-)category of linear categories with G-action to linear categories:

1. The forgetful functor, remembering the underlying category; this describes the original, pre-gauged TQFT.
2. The invariant category; this generates the gauged TQFT.

They are co-represented by the *regular*, resp. *trivial* representations of G, among categories with locally trivial action.

Theorem (Sort of)

1. *The regular representation is the Lagrangian* $\mathrm{Spec}\, H_*(\Omega G)$*.*
2. *The trivial representation is the Lagrangian* $\mathfrak{t}_{\mathbb{C}}/W = T_1^* T_{\mathbb{C}}^{\vee}/W$*.*

(I mean the categories of coherent sheaves over these Lagrangians.)

1 is clear: it describes $H_*(\Omega G)$-modules as a module category over itself.

2 is a key part to the BFM description of gauge theory.

Deformations by $H^*(BG)$ and the bulk of BFM space

The Fukaya category $\mathcal{F}(X)$ carries deformations parametrized by $H^*(X)$. The gauged category $\mathcal{F}(X)^G$ should carry deformations parametrized by $H^*_G(X)$, in particular, by $H^*(BG)$.

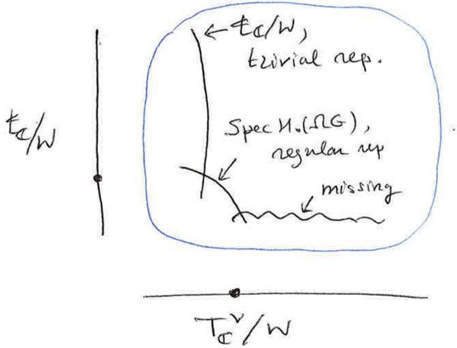
In fact, these deformations can be explained intrinsically:

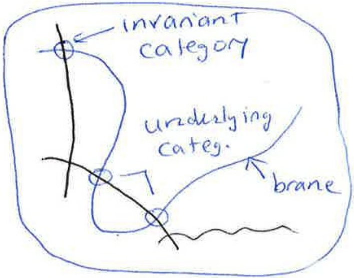
1. As TQFT deformations: an $\alpha \in H^*(BG)$ transgresses to a $t(\alpha)$ on the moduli of G-bundles over a surface. The TQFT correlator deforms by twisting the path integrand by $\exp t(\alpha)$.
2. As deformations of the G-action on $\mathcal{F}(X)$: $H^*(BG)$ parametrizes $\mathbb{Z}/2$-graded deformations of the locally trivial G-action on **Vect**. The deformed $\mathcal{F}(X)^G$ is the invariant part of the twisted category.

There is a clean geometric interpretation in the BFM space: Projection to $\mathfrak{t}_{\mathbb{C}}/W$ turns α into a Hamiltonian, and we use its flow.

The twisted representation $\mathbf{Vect}_\alpha$ is the flow of the identity fiber $\mathfrak{t}_{\mathbb{C}}/W$. This allows one to access any part of brane in the BFM bulk.

Pictures instead of thousands of words





The BFM space with the trivial and the regular representations

Invariant category and underlying category