Duality in the Ising model and its generalizations
(joint with D. Freed, Austin)

Constantin Teleman

UC Berkeley

Chicago, 7 February 2019

A Basic Problem

General Belief (Mathematician's cartoon)

$$\begin{matrix}\text{Statistical}\\ \text{mechanical models}\end{matrix} \xrightarrow[\text{limit}]{\text{continuum}} \begin{matrix}\text{Quantum Field}\\ \text{Theories}\end{matrix} \xrightarrow[\text{(if gapped)}]{\text{zero-modes}} \begin{matrix}\text{Topological}\\ \text{Field Theories}\end{matrix}$$

Phase transitions (1st or 2nd order — CFTs?) separate gapped regions

This is on reasonably sound mathematical footing in the $2D$ *Ising model*. *Continuum QFT:* the free fermion. *Two phases:* low/high temperature.

Question

Can we skip the (difficult) construction of the continuum limit above and pass straight to the classification of phases and their TQFTs?

Some progress using ideas from *extended TQFT* and *defects*:

1. Landau classification of phases by symmetry-breaking
2. Generalizing the Ising model duality to non-abelian groups

Prehistory: Ising model with group $\{\pm 1\}$

This is a "2D sigma model" from a latticed surface Σ (space-time)

- Classical fields are maps $\{\mathbf{x}_i\}$ from the vertices to the group $\{\pm 1\}$
- The action is a sum of local terms $\mathbf{x}_i \mathbf{x}_j^{-1}$ for each edge ij
- There is a global $\{\pm 1\}$ symmetry
- Local *order operators* at a node i, reading the value of $\mathbf{x}_i$
- Local *disorder operators* centered on faces (linked in pairs by paths)
- Correlators are integrals of products of operators against $\exp(-\beta S)$

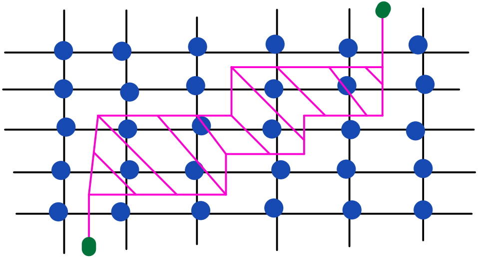
The model has a duality $\beta \leftrightarrow \beta^*$, with $\sinh 2\beta \sinh 2\beta^* = 1$, with the *dual lattice* on the *same surface*. This involves the Fourier transform on the space of fields and exchanges order/disorder operators (Kramers-Wannier).

Good textbooks will explain how this statement fails by $H^1(\Sigma; \mathbb{Z}/2)$.

The good remedy comes from *gauging the symmetry*.
This also explains the role of the linking paths (*frustration lines* in physics).

Side note: Disorder operators

Not completely local, a pair of operators must be joined by a path:



This changes the sign of each crossing edge-term in the action.
Moving the path changes the sign rules, but not the sum over all fields: sign-changing the fields swept over by the moving path relates the sums.

Non-locality is *topological*: sees the homotopy type of the frustration line
We can (and should) allow *frustration loops* too.

This applies for any target group replacing $\{\pm 1\}$.

Ising* model at the boundary of 3D gauge theory

The global ± 1 symmetry of the model can be *gauged*: we can write the model for any principal ± 1 bundle on S (double cover). The field $\mathbf{x}_i$ takes values in the fiber over i; the ratio $\mathbf{x}_i \mathbf{x}_j^{-1}$ is well-defined for each edge ij.

We only need the bundle to be defined over nodes and edges.
A monodromy (=branching) around a face is a *disorder operator*.
The frustration lines and loops fix the global topology of the bundle.

The Ising "partition function" and correlators are functions on the moduli $H^1(\Sigma; \mathbb{Z}/2)$ of bundles. Duality is fully restored by Fourier transform.

Key observation: this space of functions is the space of states for a *topologial 3D theory*, pure gauge theory with group $\{\pm 1\}$.
Kramers-Wannier duality lives on the boundary of the famous topological *electro-magnetic duality* for finite (abelian) gauge group in 3D.

Order/disorder operators are endpoints of Wilson/'t Hooft lines of electromagnetism and the two dualities are completely compatible.

Nonabelian Ising model

Electro-magnetic duality works for *arbitrary* finite gauge groups.
In the Abelian case, the dual side involves the Pontryagin dual group.
A non-abelian G has no Pontryagin dual, but has a *dual tensor category*, the G-representations, with standard $\otimes$.

Tensor categories were introduced for 3D TQFT by Turaev and Viro, but are best explained in the setting of fully extended TQFT, where the manifolds can be cut down to corners.
A tensor category with special finiteness properties (full dualizability, spherical structure) determines a TQFT on 3-manifolds by explicit combinatorial formulas.

For *group-like* categories, this is (a twist of) gauge theory. In general, it is a quantum theory with no classical model (no 'first quantization').

Novelty: One can build the Ising model on a latticed surface for certain such theories (coming from finite Hopf algebras). The Ising correlators live in the Turarev-Viro space of states for S and admit combinatorial formulas.

Idea of the construction

The Turaev-Viro space of states for a closed surface is built in two steps.

First, one constructs the space "with gauge-fixing at the nodes".
Then, one constructs a commuting family of projectors, one for each node.
The TQFT space for the closed surface is their common image.

The initial space is precisely the space of functions over the classical fields, where the Ising measure e^{-S} lives.
Projection to the TQFT space is summation over fields and gives the Ising partition function.

What we do is write a version of e^{-S} for the category of representations of a finite semi-simple Hopf algebra, given an *action function* $\beta \in H$.
Electromagnetic duality relates H with H^* and the Ising boundary theory for (H, β) with that for (H^*, β^*), where β^* relates to β by an abstract version of the finite Fourier transform.

Classification of phases by Landau symmetry-breaking

The $\{\pm 1\}$ Ising theory has a cold and a hot phase, distinguished by the vacua in the space of states for a circle (functions on the classical fields)

Cold phase: field alignment prevails. The two vacua are the δ-functions on the two classical states of constant spin. They are interchanged by the global symmetry, which is 'broken'.
Hot phase: Randomness prevails; the vacuum is the constant function on all field configurations. It is preserved by the global symmetry.

This classification by *symmetry breaking* has an analogue for a general G. One singles out a *hot subgroup* H. The action tolerates configurations where adjacent fields differ by H but penalizes jumps $\notin H$.
The vacuua are the constant functions on field configurations valued in a single H-coset; the Hamiltonian is close to the projection to the vacua.

In terms of EM theory, H defines a *partial Dirichlet boundary condition* which reduces the structure group to H on the boundary.

Justification: coupling to 3D fully extended EM theory

Any good thermodynamic or continuum limiting procedure which extracts a topological theory as a gapped sector should be compatible with the algebraic structure we spelt out.
(This is the wishful thinking part of our theorem.)

In particular, it should give a topological boundary condition for 3D EM. In the world of *extended field theories*, the simple boundary conditions are classified by subgroups H of G with a central extension.

Central extensions involve negative or complex numbers. They cannot appear from real actions whose exponentials are ≥ 0.
So the Landau symmetry-breaking subgroup classifies topological sectors.

Remark

Sums of simple theories might appear as a sector, but only at 1st-order phase transitions between gapped phases? (E.g. the $n > 4$ Potts model)
Wavy argument: perturbing the action separates the eigenvalues.

Key questions we did not answer

There is a nice space of *ferromagnetic* actions S for Ising models on G: e^{-S} is a *positive, even function on G with positive Fourier transforms*. These functions form a polytope $P \subset \mathbb{R}^{\lfloor \#G/2 \rfloor}$.

Question

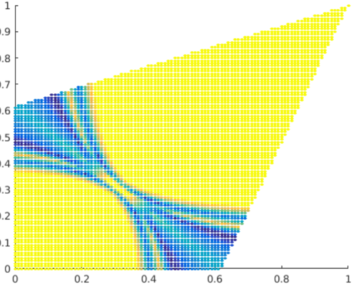
Understand the phase diagram on P.
Are the Landau regions in P connected?
(We know that each subgroup of G determines a vertex of P, with an open Landau region nearby.)
Do the other (non-subgroup) vertices lie in conformal phases?

Little is known about the phase transitions in general; speculative wisdom inspired by the XY model holds that there are open gapless regions.

Going forward, the aim is to exploit the higher algebraic structure of fully extended TQFTs to illuminate these questions.

Conjectural phase diagram for $\mathbb{Z}/5$

Basic numerical simulations suggest the following.
The two parameters are the values of e^{-S} at ω_5, ω_5^2.



Blue marks a gapless region and yellow the two Landau regions.
(The two outer blue arcs are artifacts of the limited accuracy.)
The long diagonal is the Potts line; the center of the short diagonal is the first-order phase transition.
(Computations and picture by Christoph Weiss, Oxford)