Branching of Hitchin’s Prym cover for SL(2)

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Abstract

It is shown that the map from the Jacobian of the spectral curve to the moduli of stable bundles of rank 2 is generically simply branched along an irreducible divisor. This observation falsifies the key step in the “abelianization of the SU(2) WZW connection” presented in a recent paper [Y].

1. Statement

Let Σ be a smooth complex projective curve of genus \( g \geq 3 \) and \( B \) a reduced divisor in \( |K| \). The square root \( r \) of a section of \( K \) vanishing on \( B \) defines a double cover \( p: \tilde{\Sigma} \to \Sigma \) embedded in the total space of \( K \), branched along \( B \). It is a smooth curve of genus \( \tilde{g} = 4g - 3 \), with Galois involution \( \iota \), the sign change on \( K \). \( \tilde{\Sigma} \) is the simplest example of a spectral curve [H], for rank 2 bundles on \( \Sigma \). More precisely, for a line bundle \( L \) on \( \tilde{\Sigma} \), the direct image \( E = p_*L \) is a vector bundle on \( \Sigma \), and multiplication by \( r \) on sections of \( L \) defines the Higgs field \( \phi: E \to E \otimes K \).

It is known that \( E \) is stable, if \( L \) avoids a sub-variety \( V \) of co-dimension \( \geq g - 1 \) in the Jacobian of \( \tilde{\Sigma} \) [H, BNR]. The construction works in families, so it defines a morphism \( \pi \) from the Jacobian (minus \( V \)) to the moduli space of stable vector bundles on \( \Sigma \). Moreover, \( \pi \) is generically finite, of degree \( 3^g - 3 \). We chose \( g \geq 3 \) so that singularities of the moduli spaces, as well as the stable/semi-stable distinction can be ignored.

Let us concentrate on the critical Jacobian \( \tilde{J} \) of degree \( \tilde{g} - 1 \), which maps to the moduli space \( M \) of semi-stable rank 2 bundles of slope \( g - 1 \); the story is similar for all even degrees. Call \( K^M \) the canonical bundle of \( M \). In this note, I verify the following (known) fact:

**Theorem 1.** \( \pi: \tilde{J} \setminus V \to M \) is étale away from an irreducible divisor \( D \), and is generically simply branched along \( D \). Moreover, \( O(D) = \pi^*K^M \).

Up to isogeny, \( \tilde{J} \) factors as \( J \times P \) (see 1.2) and \( D \) comes from an ample divisor on the Prym factor \( P \). The important part is the simple branching; it implies the second statement, because the canonical bundle of \( \tilde{J} \) is trivial and the Jacobian determinant of \( \pi \) gives a section of \( \pi^*K^M \) with simple vanishing along \( D \).

In a recent paper, Yoshida [Y] proposes a solution of a long-standing problem, a reduction of the flat connection in the WZW model for SU(2) to abelian Theta-functions. The key ingredient in the construction is a distinguished Theta-fuction \( \Pi \), living in a square root of the anti-canonical pull-back \( \pi^*K^M \) and vanishing along \( D \). Both properties of \( \Pi \) are essential for the constructions that follow. However, the theorem shows that such \( \Pi \) does not exist.\(^2\)

The interesting part of the story concerns SL(2) bundles and an associated Prym variety \( P \); but their relation to GL(2) is straightforward, because \( \pi \) is compatible with the tensor action (on \( J \) and \( M \)) of the degree zero Jacobian \( J \) of \( \Sigma \). More precisely, let \( K \) be the canonical bundle

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\(^1\) Yoshida constructs \( \Pi \) on an isogenous cover of \( \tilde{J} \), but the distinction is unimportant.

\(^2\) Page 2 of *loc. cit.* explicitly claims that \( \Pi^2 \) is the Jacobian determinant.
of $\tilde{\Sigma}$ and call $\tilde{B}$ the branch divisor; note the isomorphism $\tilde{K} \cong p^*K(\tilde{B})$. For a line bundle $L$ on $\tilde{\Sigma}$, the exact sequence

$$0 \to L \to p^*p_*L \to \iota^*L(-\tilde{B}) \to 0$$

shows the equivalence of the conditions

$$L \otimes \iota^*L \cong \tilde{K} \quad \text{and} \quad \det(p_*L) \cong K. \quad \text{(1.1)}$$

They define the Prym variety $P \subset \tilde{J}$. Mind, however, that the first isomorphism is always anti-invariant for $\iota$, which changes the sign on the fibres of $\tilde{K}$ over $\tilde{B}$. With $M_K$ denoting the moduli space of semi-stable bundles on $\Sigma$ with determinant $K$ and $\Gamma \subset J$ its 2-torsion subgroup, we have

$$\tilde{J} = J \times_{\Gamma} P \quad \text{and} \quad M = J \times_{\Gamma} M_K, \quad \text{(1.2)}$$

compatibly with the map $\pi$. Up to translation, the restricted morphism $P \setminus V \to M_K$ is equivalent to the Prym covering of the moduli space of $SL(2)$-bundles.

1.3 Remark. $\tilde{K} \cong \mathcal{O}(2\tilde{B})$, so one can use $L = \mathcal{O}(\tilde{B})$ to identify $\tilde{J}$ with the degree zero Jacobian; $\iota^*$ becomes an automorphism.

2. Proof

Let us abusively call the points in $\tilde{J} \setminus V$ where $\pi$ fails to be étale the ‘branch points’, even though $\pi$ is not everywhere finite; the contraction locus has high co-dimension (e.g. because the ample Theta-line bundles of the two spaces are compatible, Remark 2.5.i below). I describe the branching locus in terms of a ramified cover of a projective space and show its irreducibility. Finally, I show that the branching is simple by studying linear deformations.

(2.1) The branch locus. Let us compare first-order deformations of $L$ and of $E = p_*L$. The tangent space to $P$ is the $(-1)$-eigenspace for $\iota$ on $H^1(\tilde{\Sigma}; \mathcal{O})$, while the tangent space to $M_K$ at $E$ is $H^1(\Sigma; \text{End}^0(E))$, the traceless endomorphism bundle. Note that $p_*\mathcal{O}$ splits into the $+/-$-eigenspaces of $\iota$ as $\mathcal{O} \oplus K^\vee$, so that $TP$ is identified with $H^1(\Sigma; K^\vee)$. Unravelling the definition shows that the map induced by the Higgs field $\phi \in \text{End}^0(E) \otimes K$,

$$\phi : H^1(\Sigma; K^\vee) \to H^1(\Sigma; \text{End}^0(E)),$$

is the differential of $\iota$ at $L$. (For $\tilde{J}$ and $GL(2)$, one adds the $H^1(\Sigma; \mathcal{O})$ summands to both sides.) When $E$ is stable, both spaces have the same dimension $3g - 3$, and the short exact sequence on $\Sigma$,

$$0 \to K^\vee \xrightarrow{\phi} \text{End}^0(E) \to \Omega \to 0,$$

shows that $\pi$ is not étale iff the quotient $\Omega$ has $h^1 \neq 0$. In terms of $L$, $\Omega = p_*\left((\iota^*L^{-1}L(\tilde{B}))\right)$, and is a rank 2 vector bundle with determinant $K$. It follows from Serre duality that $h^0(\Omega) = h^1(\Omega)$. Thus, $L$ is a branch point iff $\iota^*L^{-1}L(\tilde{B})$ has sections over $\Sigma$, in other words, the last line bundle lies in the Theta-divisor $\Theta$.

(2.2) The Prym Theta-divisor. Consider the endomorphism $\sigma : L \mapsto \iota^*(L^{-1}L(\tilde{B}))$ of $\tilde{J}$. It factors via the projection to $J/J$ and lands in $P$. Restricted to $P$, $\sigma(L) = L^2(-\tilde{B})$ (or just the square, if we use $\mathcal{O}(\tilde{B})$ as base-point). We now show that $\Theta$ meets $P$ transversely in an irreducible (and locally unibranch) divisor. Its pre-image $\sigma^*(\Theta \cap P)$ will be the branching divisor $D$ of $\pi$, and we will relate transversality to simple branching.

Theta is the Abel-Jacobi image of $\text{Sym}^3\tilde{\Sigma}$, and the condition $L \otimes \iota^*L \cong \tilde{K}$ defining $P$ says that each divisor $S \in |L|$ satisfies $S + \iota(S) \in |\tilde{K}|$; multiply the matching sections of $L$ and
\( \iota^*L \). The resulting section of \( \tilde{K} \) is anti-invariant under \( \iota \), as was the isomorphism in (1.1). The anti-invariant \( p_* \)-image of \( \tilde{K} \) is \( K^2 \), and we obtain a bijection between divisors \( S + \iota(S) \in |\tilde{K}| \) and points of \( |K^2| \) (on \( \Sigma \)).

Now, \( S \) involves, in addition, a choice of point within each mirror pair in \( S + \iota(S) \). The collection of choices defines a finite cover \( \tilde{P} \) of \( |K^2| \), simply branched over the hyperplanes of sections vanishing somewhere in \( B \). The monodromy around a zero in \( B \) switches a point in \( S \) with its \( \iota \)-mirror. It is clear that the monodromies act transitively on the fibres, so that \( \tilde{P} \) is irreducible. The same follows then for the intersection \( \Theta \cap P \), which is set-theoretically the image of \( \tilde{P} \). Finally, the fibres of the Abel-Jacobi map are connected, so the image is locally unibranch.

(2.3) Simple branching. First, observe that \( P \) contains smooth points of \( \Theta \). Indeed, over a singular point \( L \in \Theta \), \( \text{Sym}^{d-1} \Sigma \) has positive-dimensional fibre, which is also the fibre of the map \( \tilde{P} \to \Theta \cap P \); but the generic fibre is finite for dimensional reasons. Next, at any smooth \( L \in \Theta \) which lies in \( P \), I claim that the normal to \( \Theta \) is a \((-1)\)-vector for \( \iota \). For this, observe that the tangent space \( T_L \Theta \) comprises the \( \xi \in H^1(\Sigma; \mathcal{O}) \) which induce the zero map \( H^0(L) \to H^1(L) \), these \( \xi \) being the first-order variations of \( L \) which carry sections. Equivalently, the co-normal line to \( \Theta \) is the image in \( T^\vee \tilde{J} = H^0(\tilde{K}) \) of the cup-product \( H^0(L) \otimes H^0(\tilde{K}L^{-1}) \). For \( L \in P \), \( \tilde{K}L^{-1} \cong \iota^*L \), so the image contains the product of a section with its \( \iota \)-transform; but we saw earlier that this lies in the \( \iota \)-anti-invariant subspace. This proves transversality.

In terms of \( \pi \), this shows that \( h^0(\Sigma; \mathcal{O}) = 1 \) generically on \( D \), and that the section fails to extend over the first-order neighbourhood of \( D \) (which surjects to that of \( \Theta \cap P \) in \( P \)). Since a first variation makes \( \phi \) an isomorphism, the branching is simple.

(2.4) Irreducibility. Finally, recall that an ample divisor on an Abelian variety of rank 2 or more is connected. As a connected étale cover of a locally unibranch divisor, \( D \) is irreducible itself.

2.5 Remark.

(i) The moduli space \( M \) is polarised by the inverse determinant of cohomology, which lifts to \( \mathcal{O}(\Theta) \) on \( J \); this is because \( H^* (\Sigma; L) = H^* (\Sigma; p_* L) \). However, \( \mathcal{O}(\Theta) \) is not principal on \( P \). One way to normalise line bundles on \( P \) is to relate them to \( M_K \), whose Picard group is \( \mathbb{Z} \). The bundle \( K_{\iota}^* \), which has Chern class 4, lifts to \( \sigma^* \mathcal{O}(\Theta) \) over \( P \). (This is the level 8 line bundle in [Y].)

(ii) The sign in §2.3 is meaningful, as the opposite would make \( \Theta \) tangent to \( P \). Now, the Jacobian determinant of \( \pi \) is the \( \bar{\partial} \)-determinant of \( \mathcal{Q} \). There is a perfect pairing \( \mathcal{Q} \otimes \mathcal{Q} \to K \), the determinant; in terms of \( \phi, q_1 \cdot q_2 \mapsto \frac{1}{2} \text{Tr} ( [\phi, q_1] \cdot q_2 ) \). The sign is in the skew-symmetry of the pairing; in the symmetric case, \( \det \bar{\partial} \) would have a Pfaffian square root.

References

