3D topological gauge theory for Lie Groups and Ungroups

1. Review (2011–)
2. Theorems
3. Sample calculus
4. Coulomb branches
5. Hypertonic calculus
6. Ungroup symmetries

1 Review (2011; 2014; etc)

Lectures and Unpublished Articles

- Oktoberfest Lectures 1, 2, 3
- Coulomb branches for disconnected groups (Edinburgh, July 2025)
- Quantizing Homotopy Types, CMSA Harvard, July 20205
- Bohr-Sommerfeld quantization revisited [Caution: some small mistakes]
- Simons Collaboration Results and Outlook (Simons Fondation Nov 24)
- Quantization commutes with reduction AGAIN (Miami HMS 2024) Notes
- Character theory for topological G-actions on categories (Miam HMS 2024) Notes
- Condensed Thoughts Simons Foundation, November 2023 Slides
- Condensation and EM duality notes (Les Diablerets, September 2022)
- 2D Quantization commutes with Reduction, Nordita, August 2023
- Gauge Theory and Mirror Symmetry in 3D
- Gromov-Witten gauge theory Lecture I and Lecture II, Bangalore, February 2023
- Coulomb branches and KO theory, MSRI, December 13, 2022
- Towards a universal target for TQFTS, Simons Foundation, November 16, 2022, Video
- Higher units in K-theory and Coulomb branches, Gregynog, July 2022
- Go and No-Go in Chern-Simons theoryMPPM Seminar Wales, June 2020
- Coulomb branches for quaternionic representations, Perimeter Institute, April 2020
- Chicago workshop, Feb 2019
- Low-dimensional topological gauge theory, Austin, Jan. 2019
- Auckland conference, Dec 2018
- The gauged symplectic sigma-model IAS Princeton, 15 November 2016
- Introduction to Quadratic Topology,YRM lecture, Oxford 2015
- Categorical representation and character theory, Berkeley Math-Physics meeting, April 2015
- Gauge theory and Mirror symmetry, ICM slides, Seoul, 2014
- Loop Groups, TQFT and Algebraic Geometry (Survey talk, MSRI, 1/31/2014)
- Five lectures on TQFT, Barcelona 2012 Errata
- Gauge theory, Langlands duality and Mirror symmetry, Simons Center, Northeastern & Columbia, Feb-April 2012 video
- Luminy version of the above, with some comments on the space of states
- Langlands duality in Mirror symmetry, Perspectives in Geometry lecture series, UT Austin, 2011
- Group actions on categories and Langlands duality (Utrecht 2011) Lecture 1 Lecture 2 Lecture 3

Long trail of lectures on topological group actions on categories
Key papers ICM 2014; Coulomb branches 2018 (also 2022)

No definitive treatment yet

Lecture: partially inspired by recent proposal (Ben-Zvi, Nadler)
of potent categorical G-representations

Will propose what may be a proper context

MetaTheorem (G connected compact Lie group)

Topological 3D, $\mathcal{N}=4$ G-gauge theory

$\Updownarrow$

RW theory of the Toda space $\mathcal{T}(G)$.

Toda space $\mathcal{T}(G)$:

- Defined by Bezrukavnikov–Finkelberg–Mirković as

  $\operatorname{Spec} H^G_*(\Omega G)$, convolution

  (also a K-theory version $\Leftrightarrow$ same for LG)

- Complitly integrable alg. group scheme / $\operatorname{Spec} H^*(BG) = \mathfrak{t}_{\mathbb{C}}/W$

- affine resolution of singularities of $(T_{\mathbb{C}}^\vee \times \mathfrak{t}_{\mathbb{C}})/W$

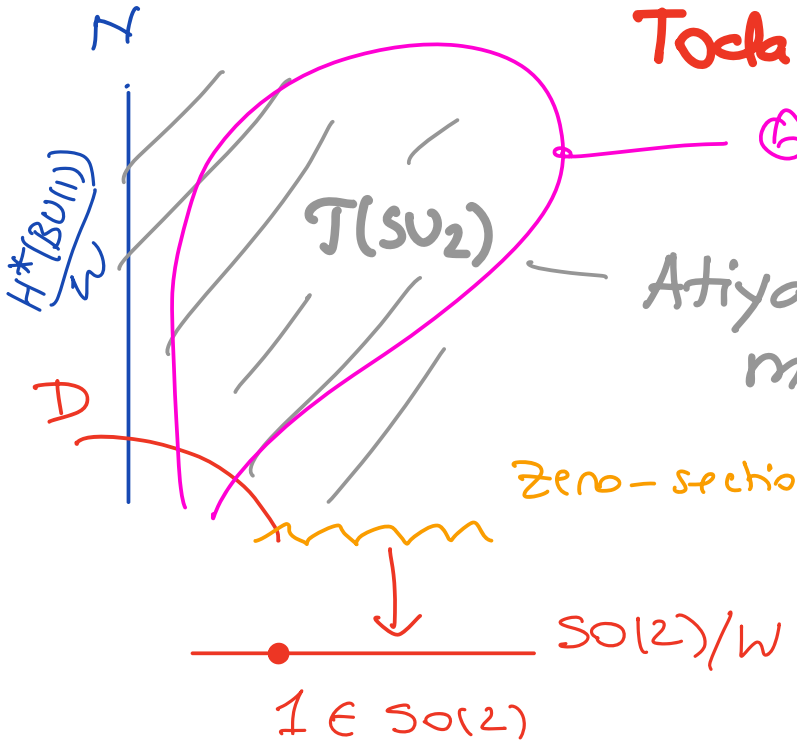
- $T^*_{\mathfrak{rs}} G_{\mathbb{C}}^\vee /\!\!/_{\mathrm{Ad}\, G_{\mathbb{C}}}$ (adjoint); $N \backslash\!\backslash T^* G_{\mathbb{C}}^\vee /\!\!/_\chi N$ (Whittaker)

  [Captures Mirrors of flag varieties)

- Poisson structure of deg. (-2) $\longrightarrow$ E_3 structure

- = Coulomb branch $\mathcal{C}(G;0)$ for the trivial rep.

- Controls Rep theory of G-categories via semiclassical calculus
  (holomorphic symplectic Lagrangians)



Toda space for SU(2)

$QH^*_{SU(2)}(\mathbb{P}^1)$ = graph of $\tau = \Sigma(z^{1/2} - z^{-1/2})$

Atiyah-Hitchin manifold

Zero-section missing (affinization)

N = Neumann boundary
D = Dirichlet boundary (gauge fixing)

Experimental Fact

All topological G-actions on categories come from 2D A-models (actions on Fukaya categories)

Thm $G \circlearrowright (X, \omega)$ Compact Hamiltonian action

$\Rightarrow QH^*_G(X)$ is an E_2 algebra central over $\mathbb{C}(G;0)$.

Thm (Pomerleano-CT) Fano (for alg. geom.)
X compact monotone $\Rightarrow QH^*_G(X) \cap N = QH^*(X/\!/G)$
(always gauged space of states $QH^*_{LG}(X)$)

Thm Fukaya category $\mathcal{F}(X)$ extends to Spec $QH^*_G(X)$
If smooth, proper (CY) over $H^*(BG)$
$\Rightarrow$ defines an object in KRS 2-category of $\mathcal{T}(G)$.

Why MetaTheorem

No all-encompassing formulation.
(No consensus complete def of either side) Weak form:

- boundary conditions match

(CY) Categories with topological G-action

⇕

Objects in KKR "2-category" of TLG)

- 3D theories are believed to be "generated by their 2 conditions."
(Not always true for 3D theories)

OK for 2-dualizable Calabi-Yau G-categories.

Without Calabi-Yau: no "convergent deformation theory".
(CY categories have unobstructed deformation theory)

History Kapustin, Rozansky, Saulina proposed a categorical calculus of holomorphic Lagrangians in hol. symplectic manifolds.

Incomplete: variants exist (eg, Miami Jan 24:
Categorified / Tropical Geometric Quantization
– Bohr-Sommerfeld method + Kontsevich formality)

Topological Action

morally: trivialization on contractible patch of the groups
Coherence of trivializations must be controlled

Eg: action on colimit of complexes bounded below
(used in work on $\mathbb{QH}^*_G(X)$ X monotone)

Curved Cartan complex: over $\mathrm{Sym}\mathfrak{g}^*$ or $\Theta(\mathfrak{g}^*)$
not just formal nbhd of 0

($S\mathfrak{g}^*$: Koszul dual of $\wedge\mathfrak{g}$ implementing
infinitesimal trivializations)

3. Sample Calculus

Answer a categorical rep theory question via characters

$SU(2)$ acts on $\mathbb{P}^1$
$SU(2)$ acts on $\mathbb{P}^1$

$SU(2) \times SU(2)$ acts on $\mathbb{P}^1 \times \mathbb{P}^1$
$\curvearrowright$ $\curvearrowleft$
$\mathbb{Z}/2$ $\mathbb{Z}/2$

So $\mathbb{Z}/2 \ltimes (N(T) \times N(T)) \circlearrowright \mathcal{F}(\mathbb{P}^1 \times \mathbb{P}^1)$
$\|$
$N(T^2 \subset Sp(2))$.

Optimistic Guess We can reduce G actions on cats to the maximal torus and the Weyl group.

Question Does the action of $\mathbb{Z}/2 \ltimes (SU(2) \times SU(2))$ on $\mathcal{F}(\mathbb{P}^1 \times \mathbb{P}^1)$ extend to $Sp(2)$?

The coordinate ring of the variety $\mathscr{T}_{SU_2}$ is

$$\mathbb{C}\left[\tau^2,\ \tau^{-1}(z-z^{-1}),\ \tau^{-2}(z+z^{-1}-2)\right] = \mathbb{C}[\tau^2,u,v]/\left(u^2=\tau^2v^2+4v\right)$$

and[1]

$$\operatorname{Spec} QH^*_{SU_2}(\mathbb{P}^1) = L_q \subset \mathscr{T}_{SU_2}, \qquad v = 4q^{-2}.$$

Therefore,

$$\operatorname{Spec} QH^*_{SU_2\times SU_2}\left(\mathbb{P}^1\times\mathbb{P}^1\right) = L_q \times L_q \subset \mathscr{T}(\mathrm{SU}(2))^2.$$

Now, the ring of functions on $\mathscr{T}_{Sp_2}$ contains additional generators on top of the $(1 \leftrightarrow 2)$-invariant part of $\mathscr{T}_{SU_2^2}$ from the short roots of Sp_2, such as[2]

$$\frac{(z_1+z_2)-(z_1^{-1}+z_2^{-1})}{\tau_1+\tau_2} + \frac{(z_1-z_2)-(z_1^{-1}-z_2^{-1})}{\tau_1-\tau_2} = 2\frac{\tau_1^2u_1-\tau_2^2u_2}{\tau_1^2-\tau_2^2},$$

which has a pole on the subvariety $L_q \times L_q$: set $q=2$, then $\tau_i^2 = u_i^2 - 4$ and get

$$\frac{\tau_1^2u_1-\tau_2^2u_2}{\tau_1^2-\tau_2^2} = \frac{u_1^3-u_2^3-4(u_1-u_2)}{u_1^2-u_2^2} = (u_1+u_2) - \frac{u_1u_2+4}{u_1+u_2}$$

blowing up along $u_1+u_2=0$, since u_1+u_2 and u_1u_2 are algebraically independent on $\mathrm{Sym}^2 L_q$.

BFM presentation becomes $\tau = dW$, $W(z) = q(z^{1/2}+z^{-1/2})$

from $(z_1^{-1}+z_2^{-1})\frac{z_1z_2-1}{\tau_1+\tau_2}$

4. Coulomb branches $\mathcal{C}(G;E)$ $E = V \oplus V^\vee$

Are to Noncompact Hamiltonian manifolds proper/V what $T(G)$ is to compact Hamiltonian G-manifolds.

Guiding belief

Rozansky-Witten theory of $\mathcal{C}(G;E)$ is

- the topological sector of 3D, G-gauge thy of E
- the G-gauged holomorphic Fukaya-Floer theory of E.

Mostly Sci-Fi, but source of predictions.

Specifically The chiral ring of point operators in ↑.

Definition proposed by Braverman-Finkelberg-Nakajima
Description simplified (CT, 118)

$V \longrightarrow$ rational Lagrangian section Σ_V of $T(G) \longrightarrow \mathfrak{t}_{\mathbb{C}}/W$

$$\Sigma_V = \begin{cases} \text{equivariant symplectic cohomology of } V \\ \text{Euler class of Atiyah index bundle of } V \text{ on } \mathbb{P}^1 \end{cases}$$

(Floer details worked out by Gonzalez-Mak-Pomerleano)

Let $R(G;E) = \mathbb{C}[G;E]$

Thm • $R(G;E)$ is the largest subring of $\mathbb{C}[T(\mathcal{O})]$ in which ϵ_V and its Toda translates are regular (CT)

• $R(G;E) \subset R(G;0)$ is the subring $\mathcal{A}_G^*(M \times V) \subset SH^*(M \times V)$, $\forall$ compact M (CT; GMP)

• Generalization to proper Kähler $X \to V$ with constraints ($\mathbb{C}^\times$-action), (Chan Chan, Lam '25)

[should work symplectically but not proved]

Consequence (Gannon-Webster; Webster)

Functoriality of Coulomb branches: $H \xrightarrow{f} G$,

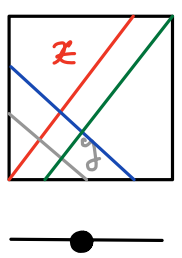
$$C(H; f^*E) = \left(C(G;E) \underset{H^*(BG)}{\times} H^*(BH) \right) / \ker f^\vee$$

(symplectic reduction)

Consequence (Bielawski-Foscolo)

Description of Coulomb branches for $G = T$ as hypertoric varieties, from

$$(\mathbb{C}^\times)^n \hookrightarrow H^n \longleftrightarrow (\mathbb{C}^2)^{\wedge n} \longrightarrow \mathbb{C}^n$$
$$(x_k, y_k) \longmapsto t_k = x_k y_k$$



Thm (CT) ("Weyl character formula"; BFN for $E=0$)

$$\mathcal{C}(G; E \oplus (\mathfrak{g}_c \oplus \mathfrak{g}_c^\vee)) \cong \mathcal{C}(T; E)/W.$$

Key Problem: Categorified version

More generally:

Describe $\mathcal{C}(G;E)$ as the $E_2 HH^*$ of the 2-category defining the 3D TQFT E/G.

(point operators)

Side Excursion "Potent representations"

(Ben-Zvi, Nadler)

- Obtained by S^1-reduction from $N=4$ in 4D
- $\Rightarrow$ Expect Langlands self-duality
- Point ops are K-theoretic $R(G; \mathfrak{g}_{IH})$
- They are NOT representations of G on category
- Closer to, but NOT representations of LG on cats
- Cannot (easily) see that $Sp(2)$ does not act on $(\mathbb{P}^1)^2$.

5 Hypertoric Calculus (Bielawski; Dancer)

By analogy with toric varieties, realize hypertorics as hol. symplectic (stacky?) reductions of $\mathbb{H}^n$ by subtori $(\mathbb{C}^\times)^k \subset (\mathbb{C}^\times)^n$.

Defined by arrangements of affine hyperplanes $\subset \mathbb{H}^n$ cut out by real equations (vectors in $(\mathbb{R}^n)^* \subset (\mathbb{H}^n)^*$).

Eg Weights of a n-dim. complex representation V of $(\mathbb{C}^\times)^k$ define such a hypertoric

need more parameters

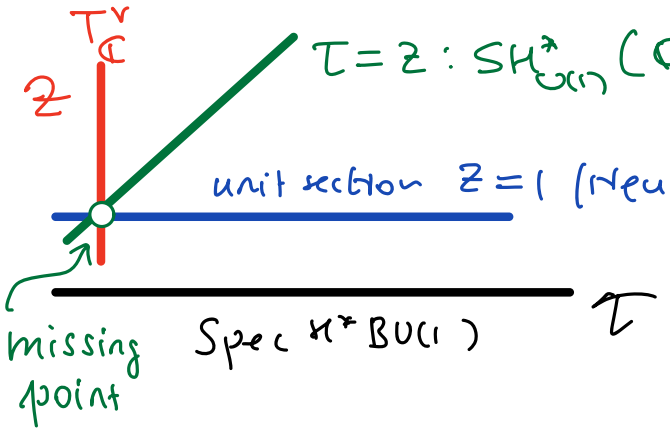
Thm (Bielawski–Foscolo) This is $\mathcal{C}((\mathbb{C}^\times)^k, V \oplus V^\vee)$

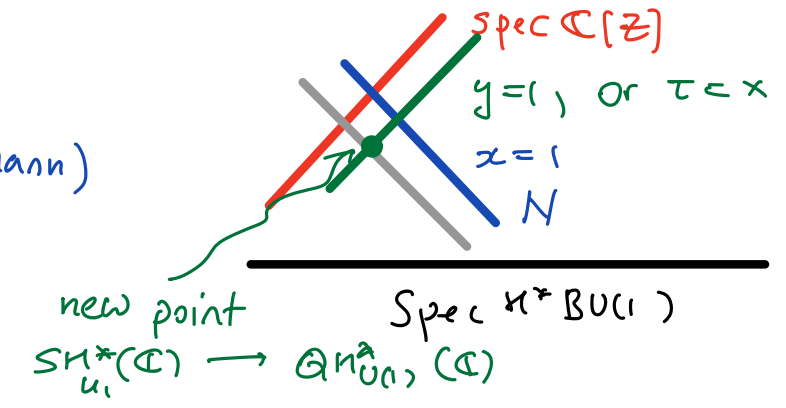
Remark The weights are set to 0, $\langle \vec{z} | \vec{v}_i \rangle = 0$ in the Coulomb branch; shifting $\langle \vec{z} | v_i \rangle = \lambda_k \in \mathbb{R}$ gives the real mass parameters, changing the complex moment map $\longrightarrow$ cx mass parameters.

Example: $U(1) \circlearrowright \mathbb{C}$, weight 1: just $\mathbb{H}$

$\mathcal{C}(T; 0) = \mathbb{C}^\times \times \mathbb{C}$ $\quad \mathcal{C}(T, \mathbb{C}^2)$: $\mathbb{C}^2 \longrightarrow \mathbb{C} = \operatorname{Spec} H^*(BU(1))$

$z, y \longmapsto \tau = zy$





6 Screwing up Lie groups

Q: What gives rise to Coulomb branches for (T, E)

A: Monoidal modifications of T:
Replace $Loc(T)$ with constructible sheaves
for a stratification defined from the stabilizers in E.

$\longrightarrow$ T-symmetries become non-invertible

Implementations

- Fukaya category of T^*T with stops
- Constructible sheaves with specified singular support

First version: described by the toric mirror symmetry recipe

Summary Insight:

Groups symmetry on categories:

implemented by $(Loc(G), *)$ as $\otimes$ category

Monoidal modification: by $(Constructible(G), *)$

w/ respect to stabilizer subgroups in the representation

[Caution with implementing multiplicities]

Toric Mirror Recipe for $U(1)$ actions

Constructible sheaves with jumps (Fukaya stops) at jumping loci of stabilizers

	A side	B side	
group	$(T^*)S^1$	$\mathbb{C}^\times = \mathrm{Spec}\,\mathbb{C}[z,z^{-1}]$	$dW = dz$
	↷ wt 1 action	↑ map, iso	$= z\cdot\frac{dz}{z}$
repns.	$\mathbb{C}$; $W=0$	$\mathbb{C}^\times$; $W=z$	$\Rightarrow \tau = z$

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group	$(T^*)S^1$, stop at 1	$\mathbb{C} = \mathrm{Spec}\,\mathbb{C}[z]$	$dW = dz$
	↷ wt 1 action	↑ iso	$t=1$ $(t=zt)$
repns.	$\mathbb{C}$, $W=z$ (1 stop)	$\mathbb{C}$, $W=z$	(cotangent)

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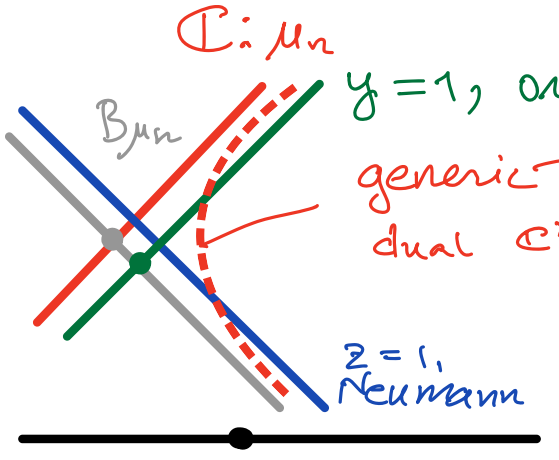
group	$(T^*)S^1$, no stops	$\mathbb{C}^\times = \mathrm{Spec}\,\mathbb{C}[z,z^{-1}]$ $W=0$	
	↷ wt. n action	↑ n-fold cover	$z=x^n$
repns.	$\mathbb{C}$, $W=0$	$\mathbb{C}^\times$, $W=x$	

---

group	$(T^*)S^1$, stops at μ_n	$\mathbb{C}:\mu_n$ stack	$dW = \frac{1}{n}z^{\frac{1}{n}}\frac{dz}{z}$
	↷ wt n action	↑ n-fold	$\tau = \frac{1}{n}z^{1/n}$
repns.	$\mathbb{C}$, $W = x = z^{1/n}$	$\mathbb{C}$, $W=x$	

$U(1) \circlearrowright \mathbb{C}$ with weight n



$\tau^n = zy$

$\tau \in \mathrm{Spec}\, H^*(BU(1))$
equiv. parameter

Summation

- Weights of representation ⟶ hypertoric data
- Multiplicities resolved by ck mass parameters
- Toric chamber defined by a polytope in hyperplane arrangement =

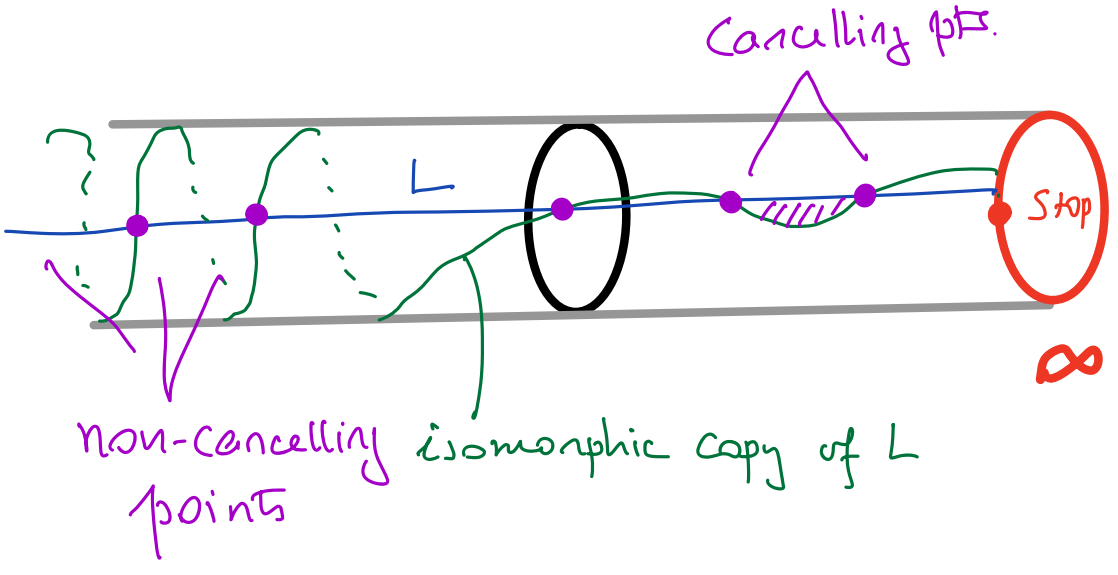
  moduli of categorical reps of stopped group
- Hypertoric stack (Coulomb branch) =

  ring of point operators for stopped group gauge theory

  $= E_2 HH^*$ of tensor category

Appendix : Ungroups

- basis of End(L)



With a stop: wrapping on one side is prohibited

$$End(L) \cong \mathbb{C}[z]$$

W/o stop: may wrap on both sides

$$End(L) \cong \mathbb{C}[z, z^{-1}]$$