Building a universal target for TQFTs

Joint work w/ Dan Freed & Claudia Scheimbauer

[Some technicalities still under verification]

Ideas have been around and exploited by many

Original project (10+ years):

Place Reshetikhin–Turaev TQFTs in the framework of the Cobordism Hypothesis $\iff$ make them fully local

Aside: manifolds have handle decompositions

(higher) categories (may) have internal dualities

handle decompositions $\xrightarrow{\text{duality data}}$ functorial TQFT operations

Cobordism hypothesis: result is independent of the handle decomposition if you can start from a point.

Reshetikhin–Turaev TQFTs are defined in dims 1, 2, 3.

$S^1 \longrightarrow$ modular tensor category $\mathcal{A}$

Sometimes, $\mathcal{A}$ = Drinfeld center of fusion category F

Then, RT theory on $\mathcal{A}$ = Turaev–Viro theory on F

= "Cob Hyp" theory on F

and is fully local.

In general, that fails and we must "invent" such F.

Enhanced project:

$\boxed{\text{Find a world for TQFTs in which all nondegenerate braided fusion cats are Drinfeld centers.}}$

$\Rightarrow$ Question

$\boxed{\text{What is a good target for (semisimple) TQFTs ?}}$

This would be a sequence ("anticategory")
$C_0 = \mathbb{C}$, $C_1 = \mathrm{Vect}$, $C_2 = \mathrm{Alg}$, ..
of $\mathbb{C}$-linear symmetric monoidal categories
with $\mathrm{End}_{C_n}(\mathbb{1}) = C_{n-1}$
a ring analogue of a spectrum in homotopy thry.

$\boxed{\text{Spectrum} = \text{sequence of groups } G_n = \Omega G_{n+1}}$

M Hopkins conjectured that the spectrum of units in the ideal anticategory should be $I_{\mathbb{C}^\times}$
$=$ Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{S}; \mathbb{C}^\times)$
(You must jump through some hoops to make precise)

$\pi_0 = \mathbb{C}^\times$, $\pi_{-1} = \mathbb{Z}/2$, $\pi_{-2} = \mathbb{Z}/2$, $\pi_{-3} = \mathbb{Z}/24$, $\pi_{-4} = \pi_{-5} = 0$

units:	even & odd line	alg. $\mathbb{C}$ & Cliff(1)	?	X
			known examples fail	

$\boxed{\text{Our results confirmed Hopkins' conjecture in dim 3 and 4 (as a toy example) and explains it in terms of symmetry } (\Sigma_d \text{ all } d).}$

Notes · Current work (Johnson-Freyd, Reutter et al) is already going beyond this in answering the Question

- Progress was made possible by work on duality in fusion and braided fusion categories (Douglas, Snyder, Brochier-Jordan et al.)
- Older work of Douglas-Henriques proposes to make Chern-Simons theories fully local via "Conformal Nets". (Requires a CFT going with the MTC, though)

Original project: Why is it interesting?
Many instances where RT theories behave like fully local ones

Gauging (Etingof-Nykshich-Ostrik; Plavnik et al; physics lit.)

$$\boxed{\begin{array}{l}\text{Gauging datum for } A \text{ by a finite group } G \\ \quad \text{homomorphism } \; G \longrightarrow (A\text{-modules}, \otimes_A)\end{array}}$$

Explanation: If $A = Z(F)$ then $A \equiv F \boxtimes F^{op}$
in such a way that $\otimes_A \longleftrightarrow \otimes_F$
So that is the datum of an action of G on F via bimodules.
Making A into $Z(X)$ would allow the same for any A.

Condensation: Condensed defects in a TV theory are exactly the surface defects.
Same goes in an RT theory for fully local defects.

Topological boundary conditions (Freed, –).
RT theories have local boundary conditions $\Leftrightarrow$ they are TV.
Need to have fully local version of RT theory to state this!

(They do allow not-fully-local boundary conds, although not semisimple ones).

Notation: Fus = Symm. Mon. 3Cat of fusion categories, bimodule cats, functors, Nat Transf.

Theorem 1 (Freed-Scheimbauer-)

Let $\mathcal{A}$ be a nondegenerate braided fusion category. Then exists a Symm. Monoidal 3-Cat. Fus$[X,Y]$ in which

- Fus is a subring and a direct summand
- $Y \cong X^\vee$
- $X \boxtimes Y \cong \mathcal{A}$
- $(\mathrm{End}(X), \circ) = (\mathcal{A}\text{-mod}, \otimes_{\mathcal{A}})$
- $\mathrm{Hom}(\mathbb{1}, X) = 0$.

In fact: there are exactly 6 of these. (Not 24...)

Theorem 2 After adding a fictitious group μ_6 of units we can do this for all such $\mathcal{A}$, incorporate the morphisms that had to be there (and end up with a symm. mon. 3Cat)

That is

$$\mathrm{Hom}(X_{\mathcal{A}}, X_{\mathcal{B}}) = \begin{cases} 0 & \text{if } \mathcal{A} \boxtimes \mathcal{B}^{rev} \neq Z(F) \text{ for any } F \\ F\text{-mod} & \text{for an iso as above, and "matching" choices of } X_{\mathcal{A}}, X_{\mathcal{B}}. \end{cases}$$

Rmk The 6 choices come from trivializing the obstruction to a symmetric monoidal structure.

Addendum With supercategories, the $\mathbb{Z}/6$ becomes $\mathbb{Z}/24$.

Onward to 4 dimensions

Fusion categories are objects in Fus
But the are morphisms in BrFus (braided fusion cats)

New object $X_A \in$ Fus $\rightsquigarrow$ new $\mathbb{1} \xrightarrow{X_A} A$ in BrFus.
This is in fact an isomorphism.
Recall that A was invertible in BrFus (Müger)
We just made it isomorphic to $\mathbb{1}$.

Recall that π_0 (Units in BrFus) = Witt group of BrFus.

(Kind of) Theorem 3.

Adjoining the new morphisms X_A to BrFus kills the Witt group and produces a symm. monoidal 4-category with End $(\mathbb{1})$ = previous 3-cat.

So we have the homotopy groups $\mathbb{C}^\times, 0, 0, \mathbb{Z}/6, 0$
Expect that with supercategories $\mathbb{C}^\times, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/24, 0$

[Some checks not finished – supercategories
– torsion in Witt group
and coherence of choices]

What is the correct 4D Category for TQFTs?

[Will hear more from Theo J.-F & collaborators]

Braided fusion categories are very special.
More generally, can consider algebra objects in Fus.

Ideas from physics (Lan-Kong-Wen)

"all 4D TQFTs should be finite gauge theories"

If we specialize to
- algebra object in Fus that are 4-dualizable ("define a TQFT")
- And are 3-dualizable as modules over themselves ("which admit a Dirichlet boundary condition")

then one can show these are finitely gauged nondegenerate BTCs (need supercats here)

Conjecture With our isomorphisms $1 \xrightarrow{x_A} A$ added in these all become gauge theories (with a Dijkgraaf-Witten class).

[Would follow if Not quite Theorem 3 clears.]

Relation to Class Field Theory

The Kronecker-Weber theorem says that
$\mathbb{Q}[\mu_\infty]$ is the universal abelian Galois extension of $\mathbb{Q}$
w/ Galois group $GL_1(\hat{\mathbb{Z}})$.

$I_{\mathbb{Q}/\mathbb{Z}}$ is a homotopical analogue of the roots of 1:

$GL_1(\mathbb{S})$ is the analogue of the Galois group
for the universal anticategory target of TQFTs.

Interesting subcategories arise as Galois fixed pts

For example, there is a subgroup

$$\begin{tikzcd}
\mathbb{S}/2 \arrow[rr] \arrow[dr] & & GL_1(\mathbb{S}) \\
& O \arrow[ur, "J"'] &
\end{tikzcd}$$

which kills η and η^2 in the sphere (the odd line and Cliff(1)) and converts $\mathbb{Z}/24$ to a $\mathbb{Z}/6$.

This possibly explains the $\mathbb{Z}/6$ we get when omitting super vector spaces.

Idea of proof of Theorem 1 illustrates the role of the symmetric group in controlling invertible TQFTs

Construction of Fus[x,y].

Let $\mathcal{A}$ be a nondegenerate BTC (invertible).

Then, $\bigoplus_{n\in\mathbb{Z}} \mathcal{A}^{\otimes n}$ is an algebra object in BrFus.

So $H := \mathrm{Hom}_{\mathrm{BrFus}}(\mathbb{1}; \bigoplus \mathcal{A}^{\otimes n})$ is a monoidal 3-cat.

containing $\mathrm{Hom}_{\mathrm{BrFus}}(\mathbb{1},\mathbb{1}) = \mathrm{Fus}$ as summand.

It also contains $X_{\mathcal{A}} := \mathcal{A} \in \mathrm{Hom}_{\mathrm{BrFus}}(\mathbb{1},\mathcal{A})$

whose endomorphism algebra is $\mathcal{A}$.

Question: Is H symmetric monoidal?

A: there's a possible obstruction and an ambiguity

$\mathcal{A}$ defines an E_∞ map $\mathbb{S}^0 \longrightarrow GL_1(\mathrm{BrFus})$.

We need to factor that through $\mathbb{Z}$.

Obstruction: the action of the fiber $\mathbb{S}^0_{>0} \hookrightarrow \mathbb{S}^0 \to \mathbb{Z}$

the action of $\mathbb{S}^{-1}_{\geq 0}$ on $\mathcal{A}$ by "change of framing"

or again the action $\Sigma_\infty \to GL(\mathcal{A})$ of the sym. mon. str.

Get a map $\mathbb{Z} \longrightarrow \mathbb{S}^1_{>1} \longrightarrow B^2GL(\mathcal{A}) = \Sigma^5\mathbb{C}^\times$ (here)

and the group $H^5(H\mathbb{Z};\mathbb{C}^\times)$ vanishes

But the group of choices is $H^4(H\mathbb{Z};\mathbb{C}^\times) = \mathbb{Z}/6$.

(Repeat for Witt group).