SCGCS Collaboration: Results and Outlook

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Simons Foundation, 14 November 2024

Who are we?

Why are we here?

Our collaboration centers on the study of

Symmetry in Quantum Field Theory

with special interest in

Symmetry Structures beyond Groups
(*Higher categories*)

and their interplay with

Topology
(*Homotopy-Categorical calculus*)

as well as implications for theory and phenomenology in

Quantum Field Theories

and the novel understanding this brings to structures in

Topology, Representation Theory, and Geometry

What are we doing?

With the support of the Simons Foundation, we have

- Organized 3 large summer graduate schools + conference combos (a fourth graduate school coming in Switzerland, 2025)
- Run a 'kick-off' meeting (2021) and 3 Annual Meetings, hosted by the Simons Foundation, preceded by smaller 'pre-meetings'
- Posted videos and arXived lecture notes from our summer events
- Co-organized/co-funded several satellite workshops and events: Aspen '22, SCGP '22, TASI '23, Nordita '23, Perimeter '24; CMSA and ICMS meetings to come in '25
- Recruited 16+ postodcs, several now in faculty positions
- Published/arXived 175+ papers
- Linked activities, papers, lecture videos at scgcs.berkeley.edu

Seriously, what ARE we doing?

Some Goals

1. Formulation of fully local topological *symmetries* and *defects*, and their coupling to general QFTs, generalizing gauge theory models
2. Developing examples of Categorical Symmetry in QFT in dim> 2
3. Higher unitarity and semi-simplicity
4. Higher Landau classification (UV-IR decomposition in phases)

Report card

1. Done: quiche calculus of defects via cobordism hypothesis
2. Mission Accomplished (yay physics team)
3. In progress: *Dagger categories* project; *Universal target of TQFT* complete to dim 4 (e.g. fusion 2-categories)
4. UV-IR symmetry constraints worked out in examples, not fully settled. (Much work by other groups, eg. Oxford, Perimeter)

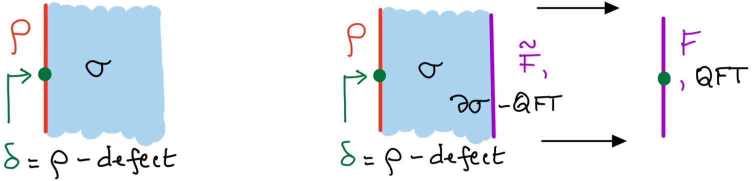
Prehistory:  Lattices, Defects, Finite homotopy

- TQFTs and defects as building blocks of *meaningful* lattice models: Kitaev&Kong '11 (Levin-Wen, Toric codes); Freed & – '18 (Ising and higher-dim. models: Electro-Magnetic/Kramers-Wannier dualities)
- *Topological condensation*: Moore & Seiberg, Bais & Slingerland, Kitaev & Kong, in low dimension (anyons); Johnson-Freyd & Gaiotto proposal in higher dimension
- (Partial) dualities in terms of defects (Fendley et al.'16)
- Obstruction to topological boundaries in self-dual TQFTs: Kapustin &Saulina; Freed&– in Reshetikhin-Turaev theories; Komargodski et al.
- *Novel TQFT Lab:* finite higher gauge theories (Kontsevich '90s)
- Proposed *fully local* construction (Freed, Hopkins, Lurie, –, '08) by successive gauging of the homotopy groups: ⇒ calculus, via Cobordism hypothesis + standard homotopy theory

Fully baked product: Quiche Calculus

Quiche: A pair (bulk symmetry theory, boundary theory) (σ^{n+1}, ρ^n): implements the (higher) algebra of symmetries $\mathrm{End}_\sigma(\rho)$ on a QFT^n.

Examples: (algebra^2, module^1); (fusion category^3, module category^2); quantized pair of finite homotopy spaces $\left(Q^{n+1}(X), Q^n(Y \xrightarrow{f} X)\right)$.



Quiche, a 'symmetry' δ and its action on a QFT F.

Many antecedents: Pure gauge theory, gaugeable boundary theory: Chern-Simons, chiral WZW; Moore-Segal open-closed TQFT; Fuchs, Runkel, Schweigert, Gaiotto-Kulp, Kapustin-Seiberg, . . .

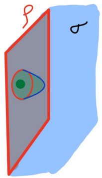
Novelty:(Freed, Moore, –) Extract topological symmetries out of QFT and use the *Cobordism Hypothesis* to execute the systematic defect calculus.

Defect Calculus: From quantized (linking) spheres and disks.

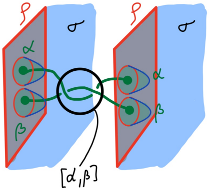
Example: σ a fusion category F, ρ a module category M,
Point defects $\leftrightarrow \mathrm{End}_F(\mathrm{Id}_M)$ (scalar, if F, M simple).

Tangential structure subtle, even when original theories are oriented.

Strong homotopy theory flavor when symmetries come from topology:



Linking disks



Braiding^2 of defects $\leftrightarrow$ Whitehead brackets.

This points to a *quantum homotopy theory* for general TQFTs.

Examples: Lattices, Dualities, Defects, Geometry

- The quiche integrates topological symmetries with lattice models of TQFT origin (many priors in dim 3/2; Freed&– more generally).
- Explicit 4d lattice theories from Fusion 2-categories (Ohmori)
- Obstruction to gapped invertible phases arise from the absence of fiber functors for the symmetry (Còrdova, Hsin, Lam et al.)
- General framework for 'twisted categorical symmetry' generalizing 't Hooft anomalies (Delaney, Freed, Plavnik, –)
- Gerby quiches can mediate between different forms of the gauge groups (change in π_1)
- Higher-dimensional examples of categorical symmetry from geometry (Bah, Del Zotto, Garcia-Etxebarria et al.): dimensional reduction from string- and M/F-theory; symmetries $\leftrightarrow$ calculus of cycles at ∞.
- 4d examples of SCFTs and SYM gauge theories with categorical symmetry (Córdova, Del Zotto, Hsin, Kaidi, Ohmori, Shao et al.)
- Much related work from groups at Oxford, Stony Brook, Penn ...

Finite homotopy (FHT): Developments and forward ideas

Foundational mathematical advances:

- Proof of functoriality of fully local FHTs via the categorical notion of *higher semi-additivity*. (Scheimbauer-Walde; under completion)
- The Chromatic Fourier transform (Schlank et al): opens the way to *EM duality* in FHTs with generalized cohomology coefficients

Guiding philosophy: close, still to be fully spelt out relation between

- the height n of an algebraic/categorical structure, as in E_n-algebras
- the chromatic height n of a cohomology theory
- the dimension $(n+1)$ of a TQFT.

Witten: dimensional reduction $\leftrightarrow$ chromatic height $+$.
Reducing a TQFT along S^1 raises the height of the ground ring:

K-theory $=$ equivariant cohomology of the loop space,
elliptic cohomology $=$ equivariant K-theory of the loop space, ...
Redshift conjecture in homotopy theory (Rognes, Burklund, Schlank et al.)

Self-dual theories in dimensions 3 (mod 4)

Invertible FHTs in $0d$ mod 4 have interesting self-dual boundary theories.

Flaw?: They usually have no topological ∂ theories of their own (Kapustin)
Plus: detect subtle invariants (Adams e-invariant; Hopkins).

Example: Modular tensor catsegories in $4d$:
from Crane-Yetter to 3d Reshetikhin-Turaev theories (Walker)

Theorem (Freed,– '20) RT theories have *fully local* topological ∂ theories
$\Leftrightarrow$ they are Turaev-Viro theories.

Theorem (Freed, Scheimbauer,–, AM'22) RT theories are fully local in an appropriate 3-category target enlarging Fusion categories.

Target has units predicted by Hopkins ($\mathbb{Z}/6$ bosonic, $\mathbb{Z}/24$ with fermions).
Arises by killing the Witt group of invertible Braided Fusion categories.

First new step in building a universal 'semi-simple' target for TQFTs:
$\mathbb{C}$, (super)vector spaces, semi-simple (super)algebras, ~~Fusion~~ cats., . . .

Classification of Fusion 2-categories

Wen, Kong, Lan Conjecture: "All 4D TQFTs are gauge theories"
(Up to invertible phases; but we now know those can be 'killed')

A mathematical source of $4D$ TQFTs are the *Fusion* 2*-categories*.
Examples: (sufficiently dualizable) *algebra objects* in Fusion categories.
Key property: Homs between objects are fusion categories.

A classification of those structures was recently completed:

Theorem (Decoppet, Johnson-Freyd, Plavnik, et al., 2024)
All $4D$ *TQFTs are gauge theories up to invertible phases:*
All fusion 2-categories are (Morita) equivalent to invertible braided tensor categories gauged by a finite group, with a Dijkgraaf-Witten twist.
(slightly generalized, in the fermonic case).

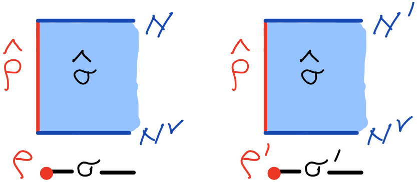
Remark: A more precise version classifies pairs (TQFT, boundary theory).
(Math language: Monoidal vs Morita classification.)

Anomalies and twisted symmetry

We know that a *'t Hooft anomaly* $\tau \in H^{n+1}(BG;\mathbb{C}^\times)$ for G-symmetry in an n-dimensional QFT F obstructs the gauging by the group G.

The *actual* symmetry σ of F is the τ-twisted G-gauge theory. This σ admits no Neumann boundary condition (*fiber functor*): $\Rightarrow$ no gauging.

DW twists can be described by *Neumann dimensional reduction*, and thus generalize to a notion of *projective* or *anomalous* symmetries:



Quiche (σ, ρ) and twisted quiche (σ', ρ'), for two variant Neumann conditions N, N' of $\hat{\sigma}$ (A related proposal: Wen '23)

Related application: the *fusion zesting* (Plavnik et al.): twists of a homomorphisms $G \to Aut(F_1)$ build twisted versions of $G \ltimes F_1$.

Interpretation (Delaney, Freed, Plavnik, –): Reductions form $4d$, with topologically twisted fiber functor over a space (built from $Z(F_1)^\times$).

Solitonic (higher?) categories

Landau Paradigm: classification of QFT phases by *symmetry-breaking*. Fine for $2d$ theories with finite symmetry group: (positive) topological boundary conditions for $3d$ G-gauge theory $\leftrightarrow$ subgroups of G.

A symmetry-related classification in $3d$ involves *superselection sectors*, long studied in Algebraic QFT, but difficult to determine.

An analogue in the (more rigorously grounded) $2d$ was recently developed by Còrdova, Ohmori, Holfester. Such a (field or lattice) theory has

- a fusion category F of symmetries
- a *category of vacua*
  (boundary conditions at ∞, governing the topological limit)

Solitons connect different vacua: this is additional dynamical information.

Theorem: The solitonic transition matrix[1] between vacua *braids with F*. This gives new relations/constraints for categorical symmetries.

[1] Matrix of vector spaces

Some Gauge Theory results

We did not neglect the more traditional group symmetries in QFT.

1. **New transitions** between *Higgs* and *confinement* SPT phases of certain 4d QCD variants (Dumitrescu & Hsin '23).
2. **Orientation/spin anomalies** for 4d gauge theory moduli spaces: (Brennan&Intriligator; Freed, Hopkins, Moore, ongoing).
3. **Quantum Reduction theorem** in 2d: (Pomerleano & – '24) Gauged QH^*(Fano symplectic space) $=$ QH^*(symplectic quotient).

   Non-Fano: Gauged $QH^* > QH^*$(quotient) explained via Floer theory. (Physics/MS literature: complex chiral ring $>$ symplectic chiral ring.)

(3) Vastly extends Batyrev's '94 toric result. Grounded in ideas from 3*d* *Mirror symmetry* (pioneered by Intriligator-Seiberg '96).
Here ($N = 4$ SUSY, 3d gauge theory), developed mathematically as equivalence to the Rozansky-Witten theory for the *Coulomb branch* (–'14).

Categories appear as *topological representations of Lie groups*: classification of 2*d* topological phases with Lie group symmetry.

Unexpected Application to Algebraic Geometry

An unforeseen use of these ideas, external to our collaboration, went into:

Theorem (announced by Katzarkov, Kontsevich, Pantev):
The generic cubic fourfold is not rational.

Final step in the proof was the QH^* *Decomposition theorem for blow-ups*:

Proved by Iritani '23 using the *Fourier Transform in gauged* QH^*, relating the J-function and equivariant Γ function for symplectic quotients.

This lives on the $3d$ Mirror model (–'14), on the Coulomb branch, and is the (trace of) *Electromagnetic duality*.
Not apocryphal: (the math translation of) this went into Iritani's proof.

Ongoing work (Iritani, Pomerleano, –): Mirror description of the phase decomposition of QH^* of quotients
(chamber decomposition of the moment polytope).

Electro-magnetically self-dual homotopy theory

Inspired by the convergence of two different regimes of homotopy theory:

1. *Rational* homotopy theory is governed by Lie algebras (Whitehead brackets on homotopy)
2. *Finite* homotopy theory governed by vastly different techniques (Cohomology operations)

Quantization brings the two closer in the calculus of defects in TQFTs.

One novel feature post-quantization is EM duality, which tries to flip the homotopy groups of a space. (Multiplicative Pontryagin/Poincaré duality.)

Continuous symmetry quantization (1) involves symplectic manifolds, Lagrangians and (partly conjectural) *higher Kapustin-Rozansky categories*. (Boundary theories for quasi-invertible TQFTs in even dimension, analogous to the self-dual theories from finite homotopy spaces.)
Quantization (2) involves spaces with limited commutativity: the ∞-catgory of boundary conditions gets truncated to an n-category.

But wait, there's more!

- Topology and SCFT: Elliptic genera and the 576-periodicity of the Hopkins spectrum of topological modular forms, in terms of proposed models for a moduli of SCFTs (Tachikawa;Yamashita,Johnson-Freyd)
- Quantization of moduli spaces of G-bundles via Skein theory: *Witten's finiteness conjecture* for modules (Jordan et al.), with application to quantum A-polynomials (Jordan&Brown, ongoing)
- Mathematical theory of defect condensation (previewed at AM '23) Beyond topology: Full WZW model is an algebra object in CFTs, condensing to Chern-Simons theory. Higher-dimensional examples?
- Universal target for TQFTs: a minimal 'higher semisimple' tower of categories, with spectrum of units the Pontryagin dual of the sphere. This includes self-dual theories in dim $4n - 1$ and their L-theory
- Higher unitarity: pursuing higher Dagger categories
- Semiclassical and Quantum FTs with corners (Cattaneo, Reshetikhin)

Where are we going?

Where are we going? Into the Unknown!

Known Unknowns

- Continuous symmetries:
  Topological ones only apply to *SUSY protected sectors* of QFTs.
  Gauging *geometric* symmetries relies on non-rigorous path integrals.
  Reconcile with the rigorous mathematics we have been practicing?
- Exotic topological phases (generalized cohomology):
  Lattice/QFT Reconciliation: do differential methods of Lagrangian QFT inform the calculus of lattice models?

Unknown unknowns

- Full locality of (non-T)QFTs. Else, what is a fully local symmetry?
- Extended non-topological operators in QFT?
  (Not built from point operators)
- Gravity: does it rule out topological symmetries?

Expect unexpected mathematics applications!

Who else is coming? (You are)

Symmetry is a proven tool to address otherwise unsolvable problems. QFT is a famous unsolvable puzzle.

Building on major developments of the last few decades in Mathematics and Physics, our Collaboration, along with numerous other researchers, showed how symmetry structures beyond groups and the accompanying higher algebra have enhanced our understanding of the structure of QFT, provided new constraints on realistic physical models, and new insights into topology, algebra and geometry.

We've witnessed overwhelming interest in these topics, and enormous contributions have been coming from the many of you who are not formally affiliated with our collaboration.

Please bring it on, and we hope to see you at our events!