C. Teleman, Lent Term 2001

Riemann Surfaces: Problem Sheet 3

Starred questions are more difficult; treat them as optional. Some questions are provided by the courtesy of Dr. A. Beardon.

Problem 1. Describe the topology of the Riemann surface $w^n - z^n = 1$ in $\mathbb{C}^2$.

Problem 2. The *cross-ratio* $\{z_1, z_2; z_3, z_4\}$ of four *distinct* points in $\mathbb{P}^1$ is the number $\frac{z_1 - z_2}{z_1 - z_4} \Big/ \frac{z_3 - z_2}{z_3 - z_4}$. One sees easily that $\{z_1, z_2; z_3, z_4\}$ can take all complex values other than $0, 1, \infty$.

(a) Two ordered quadruple of points are related by a Möbius transformation iff the cross-ratios agree. *(Hint: The equation* $\{z, z_2; z_3, z_4\} = \{z', z_2'; z_3', z_4'\}$ *defines the desired Möbius transformation* $z \mapsto z'$ *)*

(b) If $\{z_1, z_2; z_3, z_4\} = \lambda$, show that the possible values of cross-ratios under reordering the points are $\lambda, \lambda^{-1}, 1 - \lambda, (1 - \lambda)^{-1}, \lambda/(\lambda - 1)$ and $(\lambda - 1)/\lambda$.

(c) Let $\phi(\lambda) = \frac{4(\lambda^2 - \lambda + 1)^3}{27\lambda^2(\lambda - 1)^2}$. Show that two *unordered* quadruples can be transformed to each other if their cross-ratios λ, λ' (taken in any ordering) satisfy $\phi(\lambda) = \phi(\lambda')$.

(d) Let $e_{1,2} = \ (\omega_{1,2})$, $e_3 = \ (\omega_1 + \omega_2)$. If $\lambda = \{e_1, e_2; e_3, \infty\} = (e_1 - e_2)/(e_3 - e_2)$, show that

$$\phi(\lambda) = \frac{g_2^3}{g_2^3 - 27 g_3^2}.$$

Remark. It follows that two Riemann surface covers of $\mathbb{P}^1$, branched at precisely four points, are analytically isomorphic iff the two sets of branch points are related by a Möbius transform. (Cf. Lect. 9).

Problem 3. Prove the *addition theorem* for the $\ -$ function:

$$\begin{vmatrix} 1 & 1 & 1 \\ (u) & (v) & (w) \\ '(u) & '(v) & '(w) \end{vmatrix} = 0, \text{ iff: two variables agree, or } u + v + w = 0 \pmod{L}.$$

Hint: Fix v and w (not in L) and view the determinant as an elliptic function of u. The case $v + w \quad L$ *requires separate treatment.*

Problem 4. Which of the following Riemann surfaces is a "Galois cover" of $\mathbb{C}_{(z)}$? Recall that a cover $\pi: R \to S$ is Galois if there is a group Γ of automorphisms of *R*, commuting with the projection π, which acts simply transitively on the inverse images $\pi^{-1}(s)$ of most points *s*.

(a) $w^2 = 4z^3 - g_2 z - g_3$
(b) $w^n - z^n = 1$
(c)* $w^3 + z + z^2 = w^2 + wz$ *(Hint: look at the points over* $z = 0$ *).*
(d) $w^2 - 2zw + z^3 - 1$ *(Hint: Complete the square to spot the automorphism).*

Problem 5. Let R be the “hyperelliptic” Riemann surface obtained by compactifying $w^2 = f(z)$, where $f(z)$ is a polynomial of degree $2g+2$ with simple roots.

(a) Verify that the differential dz has simple zeroes over the zeroes of f, and poles of order 2 at infinity.
(b) Show that no expression $\varphi(z)dz$, with $f \in \mathbb{C}(z)$, can define a global holomorphic differential on R. (E.g, check that $\varphi(z)dz$ has poles on R over any point in the finite plane where f does).
(c) Show that a meromorphic function on R splits as $\varphi_0(z) + \varphi_1(z)/w$, with $\varphi_{0,1}(z) \in \mathbb{C}(z)$, into its even and odd parts under the automorphism $w \mapsto -w$ of R.
(d) Show that a holomorphic differential on R has the form $\varphi(z)dz/w$, where φ is a polynomial of degree less than g. Conclude that the ratios of holomorphic differentials generate the subfield $\mathbb{C}(z) \subset \mathbb{C}(R)$.

Problem 6. Let $\pi: R \to S$ be a holomorphic map of degree $d > 0$ of compact Riemann surfaces and let f be a meromorphic function on R.
(a) Show that any symmetric polynomial in the values $f(P_1), f(P_2), \ldots, f(P_d)$ of f at the d points in $\pi^{-1}(s)$ (repeated according to their valencies), as s varies over S, defines a meromorphic function on S.
Hint: Use the local form of a holomorphic map; explain, to handle higher valencies, why a meromorphic function of z, which is invariant under $z \mapsto e^{2\pi i/n} z$, *defines a meromorphic function of* z^n.
(b) Conclude from here that every meromorphic function on R satisfies a polynomial equation of degree d with coefficients in $\mathbb{C}(S)$, $f^d + p_{d-1}(s) f^{d-1} + \cdots + p_1(s) f + p_0(s) = 0$.
Hint: Use part (a), with the elementary symmetric functions.

Problem 7*. Prove Part (c) of the theorem in Lecture 12:
Let R be a compact connected Riemann surface, $\pi: R \to \mathbb{P}^1$ a branched cover of degree d, $f: R \to \mathbb{C} \cup \{\infty\}$ a meromorphic function with the property that, for some value $a \in \mathbb{P}^1$, the function f takes d distinct values at the d points of R projecting to a under π. Then every meromorphic function g on R is a rational function in z and f (z really stands for the function $z \circ \pi$).

(a) Show, using Problem 3, that

$$F_z(X) = \prod_{P \in \pi^{-1}(z)} (X - f(P)), \qquad G_z(X) := F_z(X) \cdot \sum_{P \in \pi^{-1}(z)} \frac{g(P)}{X - f(P)}$$

define polynomials in X whose coefficients are rational functions of z. (Repeated factors are needed, according to the valencies of π).
(b) For any $z_0 \in \mathbb{P}^1$ over which f and g have no poles, and $P \in \pi^{-1}(z)$, show that

$$G_{z_0}(f(P)) = g(P) \cdot F_{z_0}'(f(P)),$$

where $F_z'(X) := dF_z / dX$. (You may treat the $f(P)$ and $g(P)$ as arbitrary numbers).
(c) Show that, with P ranging over R, $F'_{z=\pi(P)}(f(P))$ defines a meromorphic function which *does not vanish identically* on R.
Hint: If, at some P, $F'_{z=\pi(P)}(f(P)) = 0$, *then* $f(P)$ *is a multiple root of* $F_z(X)$*; and then f takes fewer than d distinct values over* $\pi^{-1}(z)$.
(d) Conclude that $g(P) = G_{z(P)}(f(P)) / F'_{z(P)}(f(P))$ gives a rational expression of g in terms of z and f.

Problem 8. Recall that a group Γ acts *properly discontinuously* on a topological space X iff every $x \in X$ has a neighbourhood U whose transforms $\gamma \cdot U$, as γ ranges over Γ, are disjoint. Prove that every group of automorphisms of $\mathbb{C}$, acting properly discontinuously, is one of the following:

(i) $\{0\}$;

(ii) $\mathbb{Z}\cdot\omega$, $\omega \in \mathbb{C}^*$, acting by translations;

(iii) $\mathbb{Z}\cdot\lambda + \mathbb{Z}\cdot\mu$, $\lambda, \mu \in \mathbb{C}$ with $\lambda/\bar{\mu} \notin \mathbb{R}$, acting by translations.

Conclude that the only Riemann surfaces whose universal cover is $\mathbb{C}$ are:

$\mathbb{C}$ itself, $\mathbb{C}^*$, and the compact surfaces of genus 1.

Problem 9. Show that any holomorphic map from $\mathbb{C}$ to a compact Riemann surface of genus greater than 1 is constant.

Problem 10. (a) Show that the set of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2;\mathbb{Z})$, with a and d odd and b and c even, is a subgroup. Denote its quotient by $\{\pm 1\}$ by $\Gamma(2)$.

(b) Show that $\Gamma(2)$ acts *freely* on the upper-half plane $\mathfrak{H} := \{z \mid \operatorname{Im}(z) > 0\}$.

(c)* Show that the map $\lambda : \mathfrak{H} \to \mathbb{C}$ sending τ to $(e_1 - e_2)/(e_3 - e_2)$ is invariant under $\Gamma(2)$. Here, $e_{1,2,3}$ are the values of the $\wp$-function with periods π and $\pi\tau$ at the half-lattice points $\pi/2$, $\pi\tau/2$, $(\pi + \pi\tau)/2$.

Remark:* The map λ can be shown to be holomorphic and locally one-to-one. Moreover, it establishes a *bijection* between the quotient $\mathfrak{H}/\Gamma(2)$ and $\mathbb{C} - \{0,1\}$. This realizes $\mathfrak{H}$ as the universal covering surface of $\mathbb{C} - \{0,1\}$. With some work, we could extract a proof of these facts from what we know; for instance, by virtue of Prob. 2(c), $\phi(\lambda(\tau)) = g_2^3/(g_2^3 - 27g_3^2)$ is the "modular function $J(\tau)$" discussed in Lect. 9 (see notes), which was shown to establish a bijection of $\mathfrak{H}/SL(2;\mathbb{Z})$ with $\mathbb{C}$. There are six possible values of $\lambda(\tau)$ for a generic $J(\tau)$. Now, one can check that $\mathbb{P}SL(2;\mathbb{Z})/\Gamma(2) = S_3$, the symmetric group on three letters, and the six values of $\lambda(\tau)$, for fixed $J(\tau)$, correspond to the six orderings of the e's; etc.(See, e.g., Cohn, *Riemann Surfaces*, §4.6 and §6.3).

Analytic Extensions

Problem 11. Show that the power series

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \cdots$$

has radius of convergence 1, and that there is a dense subset S of the unit circle such that $f(rz) \to \infty$ as $r \to 1^-$, if $z \in S$. Conclude that there is no analytic extension of f outside the unit circle.

In practice, analytic continuation is rarely performed by successive Taylor expansions. Two alternative methods are illustrated below.

Problem 12. (Reflection Principle) Let f be a continuous function in the semi-disk $\{z \mid |z| < 1, \operatorname{Im}(z) \geq 0\}$, holomorphic in the interior of that region. Assume that f takes only real values on the diameter $(-1,1)$. Show

that *f* can be analytically extended to the entire disk, by defining $f(\bar{z}) = \overline{f(z)}$ when $\operatorname{Im}(z) < 0$.
Note: The result is much easier if you assume that f extends analytically a little bit below the diameter. For a star, try to do without that assumption by using contour integrals.

Problem 13. (The Γ-function) Here, you prove that the integral $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\,dt$, which converges only when $\operatorname{Re}(z) > 0$, extends to a meromorphic function of z over all of $\mathbb{C}$.

(a) Find an estimate to show that the (improper) integral $\int_0^\infty t^{z-1} e^{-t}\,dt$ converges, if $\operatorname{Re}(z) > 0$.

(b) Establish, for $\operatorname{Re}(z) > 0$, the formula $\Gamma(z+1) = z\,\Gamma(z)$. (Use integration by parts over a closed subinterval of $(0,\infty)$, and show that the boundary terms vanish in the limit).

(c) Assume that you can differentiate with respect to z under the integral sign when $\operatorname{Re}(z) > 0$ (cf. (d)). Use (b) to show that Γ extends to a meromorphic function on $\mathbb{C}$, with poles at the non-positive integers. What are the orders of the poles?

(d)* Prove that you can differentiate with respect to z under the integral sign, if $\operatorname{Re}(z) > 0$:

$$\frac{d\Gamma}{dz}(z) := \lim_{w \to z} \frac{\Gamma(w) - \Gamma(z)}{w - z} = \int_0^\infty t^{z-1} \cdot \ln t \cdot e^{-t}\,dt\,.$$

In particular, $\Gamma(z)$ is analytic in the right half-plane.

Suggestion: Show that the difference

$$\lim_{w \to z} \frac{\Gamma(w) - \Gamma(z)}{w - z} - \int_0^\infty t^{z-1} \cdot \ln t \cdot e^{-t}\,dt = \int_0^\infty \left(\frac{t^{w-z} - 1}{w - z} - \ln t \right) t^{z-1} \cdot e^{-t}\,dt$$

goes to 0 as $w \to z$ *by verifying the estimate*

$$\left| \frac{t^{w-z} - 1}{w - z} - \ln t \right| \le |w - z| \cdot (\ln t)^2 \cdot |t|^{\pm|w-z|}$$

and checking convergence of the relevant integral. The estimate is a special case of the inequality

$$\left| \frac{f(b) - f(a)}{b - a} - f'(a) \right| \le |b - a| \cdot \sup_{s \in [a,b]} |f''(s)| \qquad (*)$$

valid for a twice continuously differentiable function f on $[a,b]$ *(for instance, by a double application of Lagrange's mean-value theorem to* $\operatorname{Re} f$ *and* $\operatorname{Im} f$*). Let* $a = 0$, $b = 1$, $f(s) = t^{s(w-z)}/(w-z)$