Riemann Surfaces: Problem Sheet 1

Starred questions are more difficult; you may treat them as optional.

Refresher on complex analyis
(see also 9, 10 and 11, after you know the definitions of Riemann surfaces and holomorphic maps)

Problem 1. A differentiable map $u = u(x,y)$, $v = v(x,y)$, mapping an open set R in the (x,y)-plane bijectiveley onto an open set S in the (u,v)-plane, is called *conformal* if it preserves angles (including their signs). That means: two smooth curves C, C' meeting at (x,y) are mapped into smooth curves meeting at (u,v) with the same angle. With $z = x + i\cdot y$ and $w = u + \mathrm{i}\cdot v$, show that the map $(x,y)\mapsto(u,v)$ is conformal iff $z \mapsto w$ is analytic.
Hint: use the Cauchy-Riemann equations.
Remark: We have not assumed that the inverse map $(u,v)\mapsto(x,y)$ is diffeomorphic. To answer the question completely, try to also prove that, from the assumptions.

Problem 2. Let f be a holomorphic function in the unit disk and γ a simple closed curve inside the disk, on which f never vanishes. Let $w_1,\dots,w_n$ be the zeroes of f inside γ, repeated as their multiplicity requires. Prove, using the Cauchy formula and the local behaviour of f near a zero, that

$$\frac{1}{2\pi i}\cdot\oint_\gamma \xi^m\cdot\frac{f'(\xi)}{f(\xi)}d\xi = w_1^m + \ldots + w_n^m \ (m \geq 0).$$

Problem 3. Verify that the 2-variable power series

$$f(z,w) = \sum\nolimits_{m,n\geq 0} f_{mn}z^m\cdot w^n$$

is absolutely convergent for every pair (z,w) in the domain $D = \{(z,w)\,|\,0 \leq |z| < A, 0 \leq |w| < B\}$ if and only if, for all positive $a < A$, $b < B$, there exists a constant c such that $|f_{mn}| < c/a^m\cdot b^n$. Prove that convergence is uniform on compact subsets within D.

Conclude that convergence at some (z_1,w_1) with $|z_1| = A$, $|w_1| = B$ implies convergence in D.

Conclude that f is continuous and is holomorphic in the two variables separately. Also show that you can differentiate the series term by term, again with uniform convergence in the same regions:

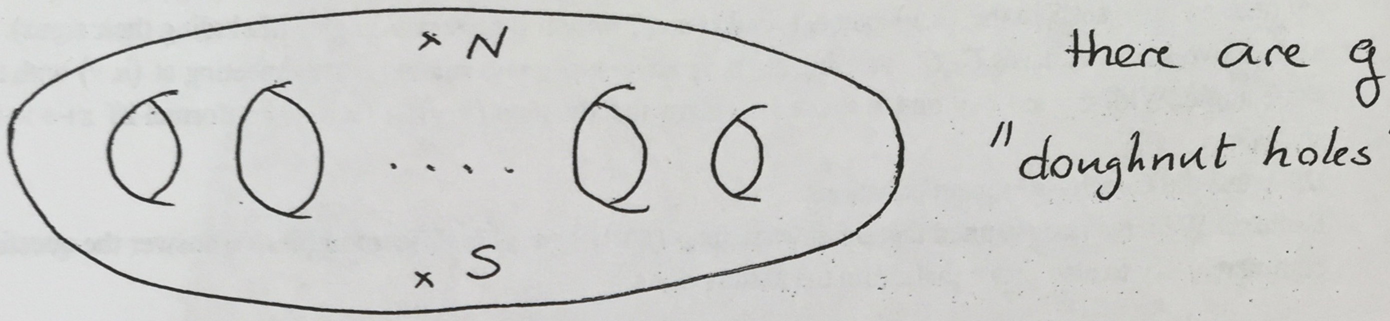
$$\partial f/\partial z(z,w) = \sum\nolimits_{m,n\geq 0} m\cdot f_{mn}z^m\cdot w^n.$$

Problem 4. Let $F(z,w)$, with $F(0,0) = 0$ and $\partial F/\partial w(0,0) \neq 0$, be a complex-analytic function of two variables in the region $\{(z,w)\in\mathbb{C}^2\,|\,0 \leq |z| < A, 0 \leq |w| < B\}$. Let S be the zero-set of F in that region. Prove that some small neighbourhood of 0 in S, lying over a sufficiently small disk $0 \leq |z| < \delta$, represents the graph of an analytic function $w = \psi(z)$, (defined for $0 \leq |z| < \delta$). You may wish to proceed as follows:

The condition $\partial F/\partial w(0,0) \neq 0$, combined with the inverse function theorem of real calculus, ensures that S represents (near 0) the graph of a continuously differentiable map $(x,y)\mapsto(u,v) = (\psi_1(x,y),\psi_2(x,y))$ (where $z = x + \mathrm{i}\cdot y$, $w = u + \mathrm{i}\cdot v$). Let $\psi = \psi_1 + \mathrm{i}\cdot\psi_2$. By differentiating the equation $F(x+iy,\psi(x,y)) \equiv 0$ with respect to x and to y, conclude that $\psi_{1,2}$ satisfy the *Cauchy-Riemann equations* $\partial\psi_1/\partial x = \partial\psi_2/\partial y$, $\partial\psi_2/\partial y = -\partial\psi_1/\partial x$ for (x,y) near 0.

Topology of Riemann surfaces

Problem 5. (a) Explain clearly how you can obtain a Riemann surface in $\mathbb{C}^2$ homeomorphic to the genus g surface with two punctures as the graph of a function $w = w(z)$, satisfying $w^2 - f(z) = 0$, for suitable f. (The surface is depicted below; the points N and S are removed)
(b)* Obtain a genus g surface with a *single* puncture, by a similar construction.
Hint: Try an f with odd degree



Problem 6. (a) *(Requires Riemann-Hurwitz)* Assume that the Riemann surface R in Problem 5(a) can be compactified, by the addition of the two points N and S, to give an abstract Riemann surface R^{cpt}, and assume that the projection of R to the z-axis extends analytically to a map from R^{cpt} to the Riemann sphere, mapping N and S to ∞. Confirm your calculation of the genus of R^{cpt} in 3(a) via the Riemann-Hurwitz formula. (You need to make sure nothing interesting happens at infinity).
(b)* Similarly, explain your result in 5(b) terms of the Riemann-Hurwitz formula, applied to the compactified surfaces. You can be informal, but should clearly explain what happens at ∞.
(c) What are the possible genera of surfaces you can produce by the above method, from $w^3 - f(z)$, when f has degree divisible by 3, and no triple roots?
Remark: If f has double roots, the surface will be singular. It turns out this does not matter, as far as topology is concerned. If you feel confident, explain why (cf. Appendix to Lecture 2).

Abstract Riemann surfaces

Problem 7. (a) Describe a "natural" Riemann surface structure on the following tori:
• the "q-torus", obtained by identifying the inner boundary of the closed annulus $1 \le |z| \le |q|$ with the outer one, via multiplication by q. (Assume $|q| \ne 1$). (Explain what you do near the boundary).
• the quotient of the vector space $\mathbb{C}$ by the lattice $L = \{m + n\tau |\ m, n \in \mathbb{Z}\}$, where $\tau \in \mathbb{C} - \mathbb{R}$ is fixed. When $\exp(2\pi i\tau) = q$, construct a bijective analytic map, with analytic inverse, between the two tori.

(b) Show, from the definition, that an open subset of an abstract Riemann surface inherits a natural structure of abstract Rieman surface.

(c) *(Gluing of Riemann surfaces)* Let U, V be open subsets of Riemann surfaces R and S. Assume given a bijective holomorphic map $\varphi: U \to V$ with holomorphic inverse. Assuming that the topological space $R \cup_\varphi S$, obtained by identifying U with V via φ, is a topological surface, show that it has a unique structure of a Riemann surface, in such a way that the Riemann surface structures induced on its open subsets R and S agree with the original ones.

(d)* Find an example of U, V, R and S as above, such that $R \cup_\varphi S$ is *not* a topological surface!

Problem 8. (a) Show carefully that the composition $g \circ f: R \to T$ of analytic maps $f: R \to S$, $g: S \to T$ be-

tween (abstract) Riemann surfaces is also analytic.
(b) Show that an analytic and bijective map $f\colon R \to S$ between Riemann surfaces is in fact *bi-analytic* (or *bi-holomorphic*): that is, the inverse map $f^{-1}\colon S \to R$ is also analytic. (f is also called an *analytic equivalence* or *conformal equivalence*).
Hint: You need to use something to get continuity, and then analyticity of the inverse!
(c) Illustrate, using the unit disk Δ and the complex plane $\mathbb{C}$, that homeomorphic Riemann surfaces need not be conformally equivalent.
(d) Show that no two of the following domains are conformally equivalent.

$$\{z \mid 1<|z|<2\}; \qquad \{z \mid 0<|z|<1\}; \qquad \{z \mid 0<|z|<\infty\}.$$

Complex analysis again

Problem 9. Let Δ be the open unit disk in $\mathbb{C}$, $\Delta^\times = \Delta - \{0\}$.
(a) Show that a bounded holomorphic function on $\Delta^\times$ extends holomorphically to all of Δ. (The singularity is removable).
(b) Show that an *injective* holomorphic function $f\colon \Delta^\times \to \mathbb{C}$ extends holomorphically over 0, or else has a simple pole there.
(Restated: injective holomorphic maps from $\Delta^\times$ to $\mathbb{P}^1$ extend holomorphically over all of Δ).
Hint: You may use the *Weierstrass-Casorati theorem*, which asserts that *any neighbourhood of an isolated essential (=non-pole) singularity of a holomorphic function has dense image in* $\mathbb{C}$.
(There is also a neat argument using Part (a) and the Jordan Curve theorem).

(c)* Improve (b) in the following way: if $f(z)=w$ never has more than N solutions in $\Delta^\times$ (N is some fixed number), then f extends holomorphically over 0, or else has a pole of order $\le N$ there.
(d)* Prove that an injective holomorphic map from $\Delta^\times$ to any compact Riemann surface extends holomorphically to Δ.
Hint for (d): You may need to use a deep theorem, conjectured by Riemann cca 1850, which says: *Every compact Riemann surface carries a non-constant analytic map to* $\mathbb{P}^1$.

Problem 10. The *automorphism group* of a Riemann surface R is the set of bi-analytic maps of R onto itself; it is a group under composition.
(a) Show that the automorphism group of the Riemann sphere is the group of *Möbius transformations*, the maps of the form $z \mapsto (az+b)/(cz+d)$, with $ad-bc \neq 0$. Show that it is isomorphic to $SL(2;\mathbb{C})/\pm 1$.
(b) Show that an injective analytic map $f\colon \mathbb{C} \to \mathbb{C}$ is of the form $f(z)=az+b$; in particular, any automorphism of $\mathbb{C}$ has that form. (Use the result in 9b)

Problem 11. (a) Show that the subgroup of $SL(2;\mathbb{C})/\pm 1$ which, under the Möbius action, takes the unit disk into itself is $PSU(1,1) := SU(1,1)/\pm 1$.
(b) Prove the Schwarz Lemma: a function f, holomorphic in the unit disk, satisfying $|f|<1$ and $f(0)=0$, must satisfy $|f(z)| \le |z|$.
(c) Prove that a bi-holomorphic map $f\colon \Delta \to \Delta$ which takes 0 to itself is a rotation of the disk.
Hint: Apply the Schwarz lemma to f and to f^{-1}, to conclude that $|f(z)|=|z|$; then use the maximum principle to conclude that $f(z)/z$ must be constant.
(d) Conclude that the automorphism group of Δ is $PSU(1,1)$.

***Problem 12.** (This refers to the optional section in Lecture 2)
Our official definition of a (possibly singular) Riemann surface in $\mathbb{C}^2$ was a subset S which was locally the zero-set of an analytic function of two variables. Recover, from here the "moral" definition — a set which represents locally the graph or a multi-valued function w of z by proving the following (weak version of the) *Weierstrass preparation theorem*:

Let $F(z,w)$ be a complex-analytic function of two variables defined in a neighbourhood of 0 in $\mathbb{C}^2$, with $F(0,0) = \partial F/\partial w(0,0) = \ldots = \partial^{n-1}F/\partial w^{n-1}(0,0) = 0$, and $\partial^n F/\partial w^n(0,0) \neq 0$. Then, there exists a function of the form $\Phi(z,w) = w^n + \varphi_{n-1}(z)\cdot w^{n-1} + \ldots + \varphi_1(z)\cdot w + \varphi_0(z)$, with the φ_k defined and analytic near $z = 0$, so that the zero-set $\Phi(z,w) = 0$ agrees with that of F, near $(0,0)$.

Do this in the following steps:
(a) Choose a δ so small that the function $F(0,w)$ has no zeroes inside $|w| \le \delta$, other than $w = 0$. Using the continuity of F, show that, for fixed, but small enough z, the function $F(z,w)$ (viewed as a function of w alone has no zeroes on the circle $|w| = \delta$.
(b) For fixed but small enough z, show that $F(z,w)$ has precisely n zeroes $w_1(z),\ldots,w_n(z)$ (with multiplicities counted as necessary) inside $|w| = \delta$. Do this by showing that the integral in Problem 2, with $m = 0$, performed over the circle $|w| = \delta$, yields a continuous integer-valued function of z (thus a constant).
(c) Using the same integrals, conclude that the elementary symmetric functions in the $w_k(z)$, etc. are holomorphic functions of z. (The individual $w_k(z)$ need not be continuous, because there is no natural ordering of the zeroes).

Parts (a)—(c) show that $\Phi(z,w) = (w - w_1(z))\cdot\ldots\cdot(w - w_n(z))$ has the desired form.