3. Central and Retractable Defects

October 12, 2026

This lecture presents some newer results relating *condensation* and *braiding* of extended operators. They were worked out this past summer by a subgroup of our Simons Collaboration[1] and are presented here (in simplified form) for the first time. This is a preliminary version.

Introduction

(0.1) Linking of defects. A notable feature of the calculus of defects (extended operators) in TQFT is their *crossing behavior*. Generalizing the case of line operators in a 3-dimensional theory $\mathcal{T}$, which define a *braided tensor structure* on the category $\mathcal{T}(S^1)$ of anyons, moving a p-fect $\mathcal{A}$ across a q-fect $\mathcal{B}$ embedded within an n-manifold M changes the value of $\mathcal{T}(M)$ when $p+q=n-1$. Higher values of $p+q$ makes the defects cross in general, and additional data is needed to define the output in any case: the defects may not be independently inserted within M. Lower values, on the other hand, allow the defects to be isotoped without crossing each other, and this does not change the value of $\mathcal{T}(M)$. (This also applies when M has boundary, if we freeze the isotopy on the boundary.) The critical case is therefore $p+q=n-1$, as in 3 dimensions: this is when defect supports can link, and want to konw how $\mathcal{T}(M)$ changes under unlinking.

When $p,q>0$, the unlinking information is captured by an automorphism of the vector space

$$(\mathcal{T};\mathcal{A},\mathcal{B})\left(S^{n-1};S^{q-1},S^{p-1}\right) \tag{0.2}$$

assigned to the unit sphere in $\mathbb{R}^n$ and its intersections with the initial and final coordinate subspaces $\mathbb{R}^q,\mathbb{R}^p$, carrying the defects $\mathcal{A},\mathcal{B}$. To get a valid TQFT picture inside the sphere, and produce a vector in (0.2), those subspaces must be slightly displaced from each other at the origin: this is done by bumping them slightly, in the two possible ways, along the additional $(q+1)$-coordinate. The two output vectors in the space (0.2) are related by an automorphism, defined by moving one sphere around the other within S^{n-1}.

0.3 Remark. The familiar case $n=3$ and $p=q=1$ is partly misleading: the spheres S^0 are disconnected. We can then reinterpret the picture above as giving a braiding isomorphism

$$\mathcal{A}(pt)*\mathcal{B}(pt)\xrightarrow{\sim}\mathcal{B}(pt)*\mathcal{A}(pt)$$

in the category $\mathcal{T}(S^1)$ associated to the equator of the sphere S^2, placed to separate the incoming and outgoing trajectories of $\mathcal{A}$ and $\mathcal{B}$.

The case of a point and a hypersuface is also special, but easier: the hypersurface defect gives an endomorphism of $\mathcal{T}$, and we are asking how that acts on point operators (as they cross the hypersurface). Our standard assumption is that $\mathcal{T}$ is simple, so that point operators are scalar and are then unaffected by the crossing. We can say that point defects are central in a simple TQFT.

[1] C. Cordova, D. Freed, M. Hopkins, G. Moore, and myself. We thank the Aspen Physics Centre for its hospitality, which enabled this work.

0.4 Definition. A p-fect is *central* if it *commutes locally* with all q-fects: that is, the output of $\mathcal{T}$ is unchanged upon its crossing any q-fect, irrespective of any other defects present in M that are supported away from the crossing point.

(0.5) Condensation. Last time, we reviewed the important notion of *condensation* of topological defects (extended operators), whereby certain defects with an algebra structure may aggregate, or condense multiple copies of itself into a defect of one higher dimension. (For more material, see also Condensed Thoughts.) In relation to the crossing problem: if a p-fect is condensed from a $(p+1)$-fect, it seems clear that it must be central: this is the familiar 'ripping open and closing back up' argument in physics. (Once the p-fect is open, its support may be dimensionally reduced and it can freely slide around the q-fect. It's not difficult to make this precise, as we will see.) An oft-unspoken assumption in the literature is a *converse*:

0.6 Conjecture. *Central defects are condensed.*

This would be an excellent characterization of condensed defects, avoiding the need to identify a condenser. However, like all bold conjectures, it must be made more precise and corrected. A standard counterexample takes $\mathcal{T}$ to be the trivial TQFT and the p-fect $\mathcal{A}$ any q-dimensional TQFT admitting *no topological boundary theories*; examples for $q = 3$ are most Reshetikhin-Turaev theories. The defect $\mathcal{A}$ is obviously central, as it does not interact with any p-dimensional TQFT $\mathcal{B}$ embedded in $\mathcal{T}$, and is not generated by an algebra object in 2-dimensional TQFTs (which would be a semi-simple tensor category).

(0.7) Locality of defects. The first clarification concerns the *locality type* of defects considered, the extremes being fully local and global. Defects could be thought of as TQFTs embedded within $\mathcal{T}$ which are local in that they may be cut and reglued up to a specified co-dimension (not exceeding the locality co-dimension of $\mathcal{T}$, nor of course the maximal dimension of their support). The *tier count* of the theory, or defect, is one higher than this co-dimension, matching the categorical depth of the cutting and gluing structure.

Fully local defects have maximal tier. *Global defects* do not quite fit the embedded TQFT model: their supports may still be cut and reglued, with tier count one lower than $\mathcal{T}$, but they may only live on a restricted category of submanifolds. For instance, in a 3-tier $\mathcal{T}$, a global p-fect on Z is a vector in $\mathcal{T}(Z \times S^p)$, and can be factored under cutting Z along a hypersurface.

Armed with this clarification, we can formulate several versions of the conjecture.

(i) $\mathcal{T}$ is a 3-*tier TQFT* with *global defects*: $\mathcal{T}$ is valued in the 2-category of $\mathbb{C}$-linear, finite semi-simple categories, functors and natural transformations.

We can then prove the conjecture with the *condensed* property replaced by the weaker *retractability* condition below.

(ii) $\mathcal{T}$ is a fuly local, finite homotopy TQFT, and the defects (also fully local) are obtained by quantizing finite homotopy spaces.

We can then confirm the conjecture in full, from the classification of condensed defects and a linking calculus that we can explicitly execute.

(iii) $\mathcal{T}$ is a fully local TQFTs with fully local defects, and we weaken the conclusion to a *generic decoupling from* $\mathcal{T}$: a central defect $\mathcal{A}$ can interface to a constant TQFT $\mathcal{A}'$ (decoupled from $\mathcal{T}$), and $\mathcal{A}$ is reconstructible from this interface.

While we know ways to make this last propery precise, we only know how to prove the conclusion when $\mathcal{T}$ itself is a finite homotopy TQFT.

The arguments for (ii) and (iii) are more involved, as concerning algebra objects and retracts in higher categories; I will not review them here. Note that (ii) and (iii) are stronger in some ways, and weaker in others, than (i); the blanket assumption of full locality is very strong, but on the flip side, centrality with respect to fully local q-fects is a weaker assumption.

1. Retractability

Let $\mathcal{T}$ be a 3-tier TQFT as advertised, with global embedded defects. Our working substitute for the property of being condensed is

1.1 Definition (Retractability)**.** A p-fect $\mathcal{A}$ is *retractable* (or 1-retractable, retractable to codimension 1) iff any insertion $\mathcal{A}$ on a q-submanifold $Z \hookrightarrow M$, with any additional extended operators inserted, produces the same correlators as a defect $\mathcal{A}'$ supported on any given prescribed $(q-1)$-skeleton $Z' \subset Z$, onto which all additional defects that cross Z have also been retracted.

This seems like a fine operational definition, but needs clarification. Defects meeting Z need additional data to be well defined when $\mathcal{A}$ is also placed on Z. How is that data to convert to intersection data with $\mathcal{A}'$ on Z'? The only functorial, local way to do so is to posit an *interface* $\mathcal{I}$ between $\mathcal{A}$ and the transparent defect (an end of $\mathcal{A}$) so that excising a disk from Z and placing $\mathcal{I}$ on the boundary produces the same correlators. Then, any other defects may be drawn into the skeleton $Z' \subset Z$ as the disk expands to fill a top cell.

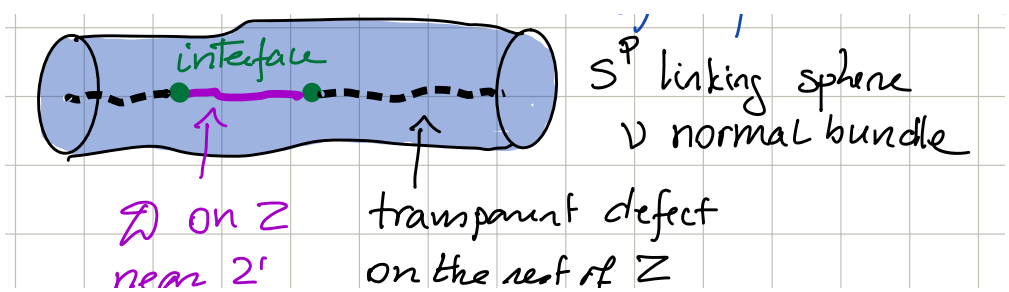


Figure 1: $\mathcal{A}$ on Z, near $Z' \subset Z$ (purple); the transparent defect on the rest of Z (dashed); the interface (green). S^p is the linking sphere in the normal bundle.

1.2 Definition (Retractability defined properly)**.** A p-fect $\mathcal{A}$ is *retractable* iff there exists an end $\mathcal{I}$ defined on spheres S^{q-1} so that removing disks D^q from Z and coloring the new boundaries ∂Z with $\mathcal{I}$ does not change the correlators, irrespective of any other insertions away from D^q.

The retraction of $(\mathcal{A}, \mathcal{I})$ onto a small tubular neighborhood of $Z' \subset Z \setminus D^q$ defines $\mathcal{A}'$.

For convenience, we reduce on S^p, converting the picture to the one in Figure 2, with

$$\mathcal{T}' = \mathcal{T}/S^p := \mathcal{T} \text{ reduced on } S^p,$$

a theory of dimension $q+1$, and its boundary theory $\mathcal{T}/D^{p+1}$.

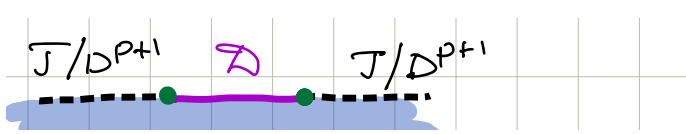


Figure 2: After reduction on S^p: boundary condition $\mathcal{T}/D^{p+1}$ on either side of $\mathcal{A}$.

We now rephrase the definition into a TQFT diagram before translating algebraically:

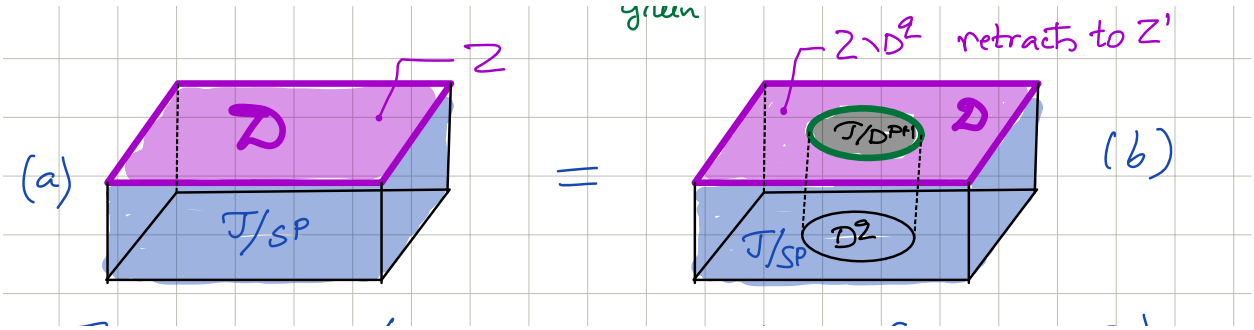


Figure 3: Left (a): $\mathcal{D}$ on all of Z on the top face, $\mathcal{T}/S^p$ in the bulk. Right (b): $\mathcal{T}/D^{p+1}$ on D^q and $\mathcal{D}$ on $Z \setminus D^q$, which retracts to Z'.

With the normal disk bundle to Z denoted by ν, we require the equality of

$$\mathcal{T}(\nu, Z, \mathcal{A}) = (\mathcal{T}/S^p,\ \mathcal{A})\,(Z \times I,\ Z \times \{0\}) \tag{a}$$

$$\mathcal{T}(\nu, Z', \mathcal{A}') = (\mathcal{T}/S^p,\ \mathcal{A} \sqcup_{\mathcal{I}} \mathcal{T}/D^{p+1})\,(Z \times I,\ Z \times \{0\}) \tag{b}$$

We compute both (a) and (b) by cutting out and reinserting the cork $D^q \times I$. For figure (a), the top boundary of the cork is still $\mathcal{A}$ and the interface is Id.

In the category $\mathcal{T}'(S^{q-1})$ we have the (co-)objects

$$\mathcal{T}'(Z \setminus D^q)^{\mathrm{op}} \ \in\ \mathcal{T}'(S^{q-1}) \ \ni\ \mathcal{T}'(D^q),$$

and the vector space of morphisms

$$\mathcal{T}(S^p \times Z) = \mathcal{T}'(Z) \ =\ \mathrm{Hom}_{\mathcal{T}'(S^{q-1})}\big(\mathcal{T}'(Z \setminus D^q),\ \mathcal{T}'(D^q)\big)$$

factored through $\mathcal{A}(S^{q-1})$ by the outer, curved arrows in the diagram

$$\begin{tikzcd}
 & \mathcal{A}(S^{q-1}) \arrow[d, dashed, "\mathcal{I}\ \text{interface}"] \arrow[ddl, bend right, "\varphi_{Z\setminus D^q}"'] \arrow[ddr, bend left, "\varphi_{\mathcal{A}}=(\mathcal{T};\mathcal{A})(D^{p+1}\times D^q;\{0\}\times D^q)"] & \\
 & \mathcal{T}(D^{p+1} \times S^{q-1}) \arrow[dr, "\psi_{\mathcal{A}}= \mathcal{T}(D^{p+1}\times D^q)"] & \\
\mathcal{T}'(Z \setminus D^q)^{\mathrm{op}} \arrow[r, phantom, "\in"] & \mathcal{T}'(S^{q-1}) \arrow[r, phantom, "\ni"] & \mathcal{T}'(D^q) \\
\text{(co-object)} & \text{(category)} & \text{(object)}
\end{tikzcd}$$

The defect $\mathcal{A}$ gives an object $\mathcal{A}(S^{q-1}) \in \mathcal{T}'(S^{q-1})$, together with morphisms

$$\varphi_{\mathcal{A}}(Z \setminus D^q) := (\mathcal{T}', \mathcal{A})\big((Z \setminus D^q) \times I\big), \qquad \varphi_{\mathcal{A}}(D^q) := (\mathcal{T}', \mathcal{A})\big(D^q \times I\big),$$
$$\mathcal{I}(S^{q-1}) : \mathcal{A}(S^{q-1}) \longrightarrow \mathcal{T}'(S^{q-1}) \quad \text{(the interface)},$$
$$\psi_{\mathcal{A}} := \mathcal{T}\big(D^{p+1} \times D^q\big) \quad \text{(the surgery morphism)}.$$

Then

$$\mathcal{A}(Z) = \varphi_{\mathcal{A}}(D^q) \circ \varphi_{\mathcal{A}}(Z \setminus D^q) \qquad \in \mathrm{Hom}_{\mathcal{T}'(S^{q-1})}\big(\mathcal{T}'(Z \setminus D^q),\ \mathcal{T}'(D^q)\big), \tag{a}$$

$$\mathcal{A}'(Z) = \psi_{\mathcal{A}} \circ \mathcal{I} \circ \varphi_{\mathcal{A}}(Z \setminus D^q) \qquad \in \mathrm{Hom}_{\mathcal{T}'(S^{q-1})}\big(\mathcal{T}'(Z \setminus D^q),\ \mathcal{T}'(D^q)\big). \tag{b}$$

1.3 Theorem. *The p-fect $\mathcal{A}$ is retractable iff there exists $\mathcal{I} \in \operatorname{Mor} \mathcal{T}'(S^{q-1})$ such that the triangle*

$$\begin{tikzcd}
\mathcal{A}(S^{q-1}) \arrow[rr, dashed, "\mathcal{I}"] \arrow[dr, "\varphi_{\mathcal{A}}(D^q)"'] & & \mathcal{T}(D^{p+1} \times S^{q-1}) \arrow[dl, "surgery"] \\
& \mathcal{T}(S^p \times D^q) &
\end{tikzcd}$$

commutes.

Proof. The vectors (a) and (b) are the compositions $\varphi_A(D^q) \circ \varphi_{\mathcal{A}}(Z \setminus D^q)^\vee$ and $\psi_A \circ \mathcal{I} \circ \varphi_{\mathcal{A}}(Z \setminus D^q)^\vee$, and must agree for any other field insertions in M away from $D^q \times I$, which isonly possible if the triangle above commutes. $\square$

2. Central defects

Here, $\mathcal{T}$ is an arbitrary TQFT of dimension $n = p+q+1$, $\mathcal{A}, \mathcal{B}$ are a p-fect and q-fect, respectively. In a top-dimenisional M, they will have supports of dimensions q and p, respectively. The datum of the defect is defined from boundary conditions for the reduced theories $\mathcal{T}/S^p, \mathcal{T}^/S^q$, or interfaces

$$\mathcal{A} : \mathbf{1}^n/S^p \longrightarrow \mathcal{T}/S^p, \qquad \mathcal{B} : \mathbf{1}^n/S^q \longrightarrow \mathcal{T}/S^q,$$

whose locality, or tier count should be prescribed (not to exceed that of $\mathcal{T}$, or their dimensions). Global defects are defined using a spherical neighborhood $\partial\nu$ of their support Z: a morphism $\mathcal{T}(\emptyset) \to \mathcal{T}(\partial\nu)$, representing tubular neighborhood with the embedded defect

(2.1) Linking and unlinking. We investigate their linking behavior in the top dimension, specifically when we have an equality:

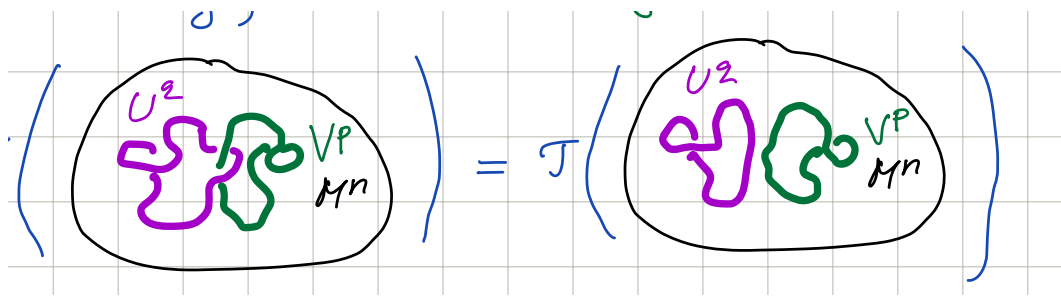


Figure 4: $\mathcal{A}$ (purple) and $\mathcal{B}$ (green) in M^n: linked = unlinked.

Centrality of $\mathcal{A}$ requires unliking to hold *universally*, for all q-fects $\mathcal{B}$, with arbitrary additional defects embedded in M and allowed to meet the supports of $\mathcal{A}$ and $\mathcal{B}$ away from the locus of the unlinking move:

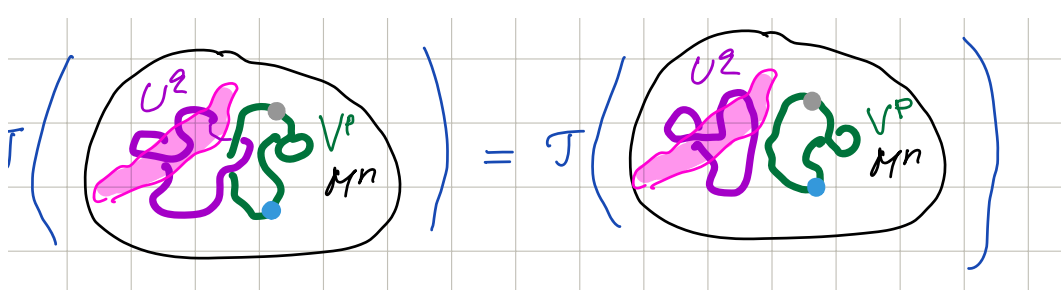


Figure 5: The universal version: additional defects (shaded) may meet the two spheres.

(2.2) Central defects. Now, we assume that $\mathcal{T}$ is simple (the space of point operators is $\mathbb{C}$), and prove Version (i) of the Unlinking Conjecture 0.6.

2.3 Theorem. *In a 3-tier TQFT $\mathcal{T}$ with global defects, Retractable $\iff$ Central.*

Proof. This is because the condition in Theorem 1.3 matches the one in Theorem 2.4 below. $\square$

2.4 Theorem. *$\mathcal{A}$ is central iff, in the category $\mathcal{T}(S^{q-1} \times S^p)$, a factorization exists:*

$$\begin{tikzcd}
(\mathcal{T},\mathcal{A})\big(D^{p+1} \times S^{q-1},\ 0 \times S^{q-1}\big) \arrow[rr, "S(\mathcal{A})"] \arrow[dr, dashed, "\exists"'] & & \mathcal{T}(S^p \times D^q) \\
& \mathcal{T}(D^{p+1} \times S^{q-1}) \arrow[ur, "S(\mathbf{1})"'] &
\end{tikzcd}$$

Here $S(\mathcal{A})$ is the defective surgery *map (surgery on the $D^{p+1} \times S^{q-1}$ containing the defect $\mathcal{A}$ along $\{0\} \times S^{q-1}$) and $S(\mathbf{1})$ is the ordinary surgery map.*

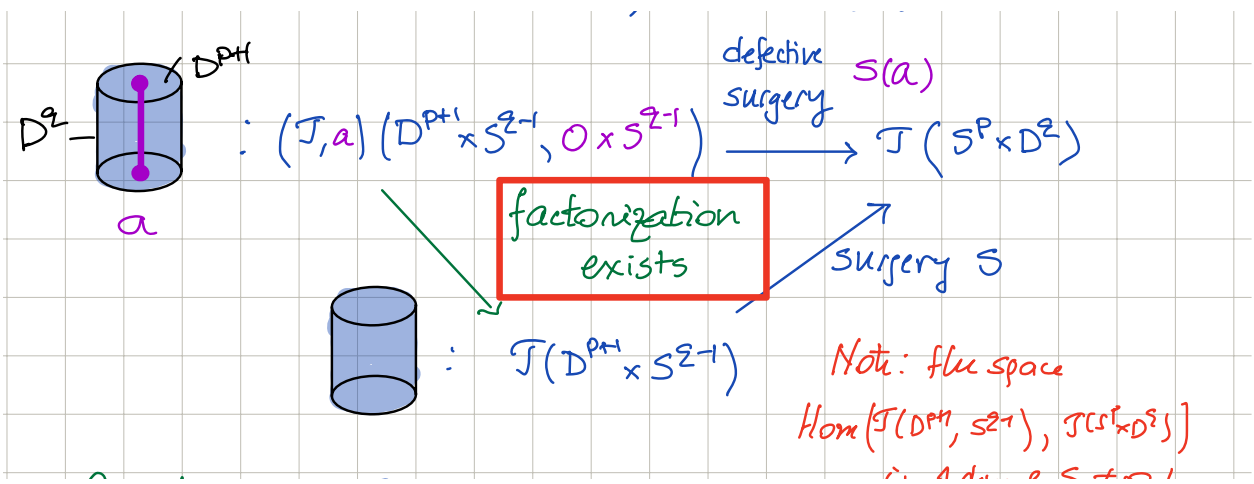


Figure 6: Theorem 2.4 in the original: the defect a (purple) sits on $0 \times S^{q-1} \subset D^{p+1} \times S^{q-1}$.

2.5 Remark. The category $\mathcal{T}\big(S^p \times S^{q-1}\big)$ has two commuting multiplications, from the two sphere factors. The objects $\mathcal{T}\big(D^{p+1} \times S^{q-1}\big)$ and $\mathcal{T}(S^p \times D^q)$ are the two units. The space

$$\mathrm{Hom}\big(\mathcal{T}(D^{p+1} \times S^{q-1}),\ \mathcal{T}(S^p \times D^q)\big)$$

is one-dimensional, being identifiable with the space $\mathcal{T}(S^{n-1})$. In particular, $S \neq 0$: it identifies the unique common projector in the two units.

Proof of Theorem 2.4. Make the (global, at first) q-fect $\mathcal{B}$ a linked S^p:

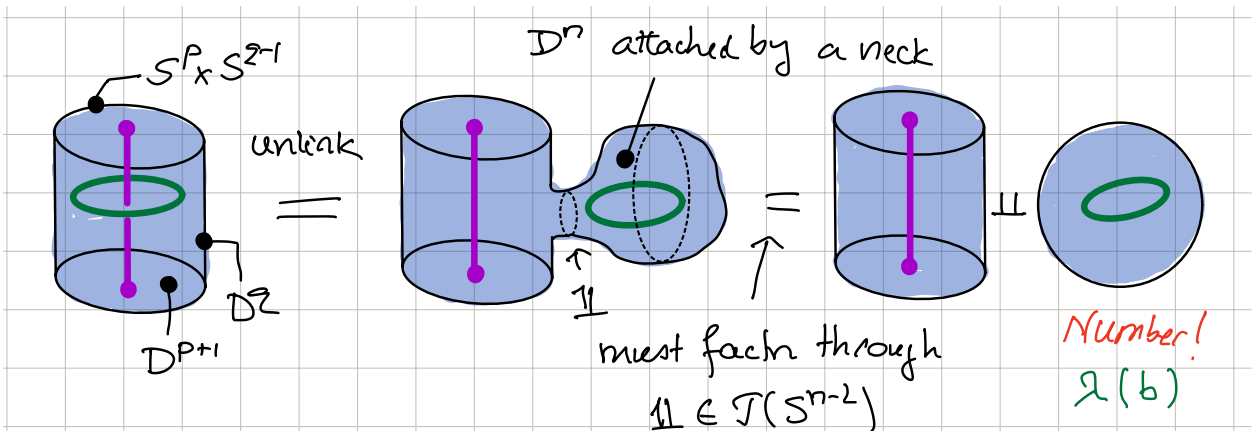
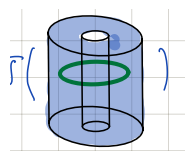


Figure 7: Unlinking b from a. The neck must factor through $\mathbf{1} \in \mathcal{T}(S^{n-2})$, so the right-hand factor is a number $\lambda(b)$.

Define $b* \in \operatorname{End} \mathcal{T}(S^p \times D^q)$ as $\mathcal{T}$ of the cored cylinder $(S^p \times [1/2, 1]) \times D^q$ with $\mathcal{B}$ inserted:

$b* \;=\; \mathcal{T}\Big($  $\Big)$.

The non-braiding condition is then

$$(b*) \circ S(\mathcal{A}) \;=\; \lambda(b) \cdot S(\mathcal{A}).$$

Now reduce on S^p to $\mathcal{T}' := \mathcal{T}/S^p$. Then $\mathcal{T}'(D^q) \in \mathcal{T}'(S^{q-1})$ is the unit for multiplication, and

$$\mathcal{T}'(D^q) = \sum_i \pi_i, \qquad \text{commuting projectors.}$$

Any point operator in $\mathcal{T}'$ comes from some global q-fect $\mathcal{B}$ on the sphere S^p. Because $S(\mathcal{A})$ is an eigenvector for all point operators in $\mathcal{T}'$, it must map into a single, simple projector.

The transparent defect $\mathbf{1}$ is also central; then so is $\mathbf{1} \oplus \mathcal{A}$. So $S(\mathcal{A})$ lands in the same projector as the transparent defect: whence we conclude the factorization. The implication retractable $\Rightarrow$ central is the obvious one. $\square$

(2.6) The unlinking criterion We conclude with a criterion for unlinking two defects in terms of the output in $\mathcal{T}(S^{n-1})$ of the Hopf link in the n-disk D^n. A sample defect that can be embedded is built from two spheres joined by a line defect x; the two spheres may then be linked, or unlinked:

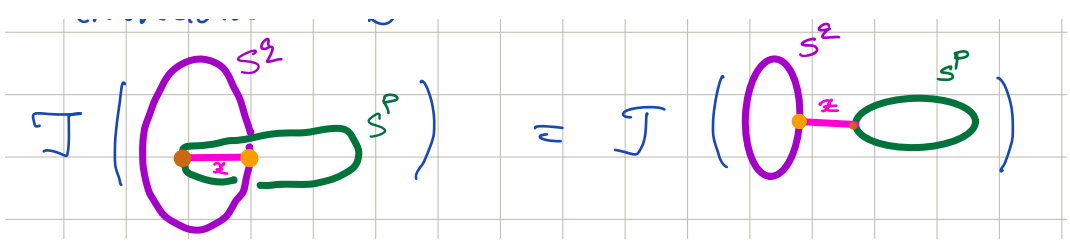


Figure 8: $(\mathcal{T}; \mathcal{A}, \mathcal{B})\big(S^q, S^p \text{ linked, joined by } x\big) = (\mathcal{T}; \mathcal{A}, \mathcal{B})\big(S^q, S^p \text{ unlinked, joined by } x\big)$.

Clearly, the agreement of the two outputs is a *necessary* condition for generally unlinking $\mathcal{A}, \mathcal{B}$.

2.7 Theorem. *When $\mathcal{T}$ is simple, this is the universal example for testing the general unlinking of $\mathcal{A}$ and $\mathcal{B}$. Moreover, the criterion reduces to a product formula for the Hopf link (when $x = \mathbf{1}$), and the vanishing of the left-hand picture for simple $x \neq \mathbf{1}$.*

Proof. We reduce a general unlinking to the one depicted in two steps. The first is displayed in Figure 9,

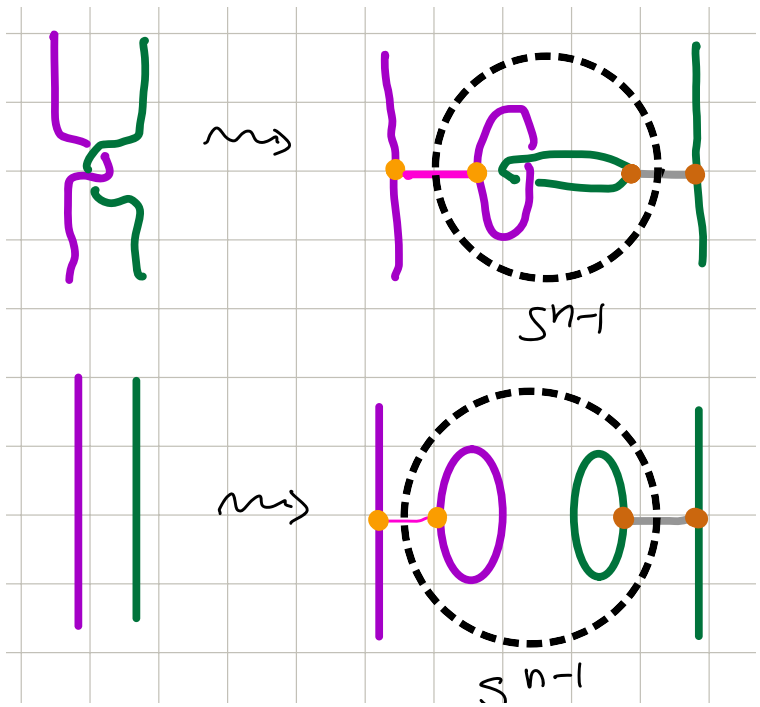
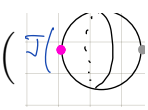


Figure 9: Braided and unbraided strands, after isolating a ball bounded by S^{n-1}.

following which we cut out the ball enclosing the Hopf link. The two output vectors must then agree in the space for the sphere S^{n-1} with the two marked insertions points (pink ang grey):

$\mathcal{T}\Big($  $\Big)$.

The insertions are objects in the category $\mathcal{T}(S^{n-2})$; decomposint into simples, the space is zero if the two insertions are not dual; whereas if they equal x and $x^\vee$, we can cancel them on the sphere by drawing tehm into a cord labelled by $x$joining the two internal defect spheres.

The right-hand picture in Figure 8, with the unlinked sphere, may now be split by isolating the spheres into two balls with a single x-marked point on their boundary: the space for each boundary sphere is then zero if $x \neq \mathbf{1}$. $\square$