Bohr-Sommerfeld Quantization

(Revisited)

[Kapustin, Rozanski, Saulina (2009)]

Proposed a 2-category associated to a holomorphic symplectic manifold X intended to classify the topological ∂ conditions for the Rozansky-Witten 3D TQFT of X.

Relevance to Mirror Symmetry: (3D, base case)

KRS[Toda space of $G^{\vee}_{\mathbb{C}}$] $\Longleftrightarrow$ 2-cat. of 2D TQFTs with G gauge symmetry

amenable to math definitions

Only partial definitions [Special cases or special objects inside]

Background:

Rozansky-Witten (1990's) defined invariants of certain 3-manifolds, expected to come from a 3D TQFT. (Partially defined? Not settled.)

That is a space analogue of Reshetikhin-Turaev TQFT.

	RW	Chern-Simons thry
Source	Hol. symplectic X	stack BG, $h \in H^4(BG; \mathbb{Z})$
S^1	$D^b Coh(X)$, braided [Kapranov].	$Rep\, U_{e^{2\pi i/h}}(\mathfrak{g})$ (semisimplified)
Σ_g	$H^*(X; (\Lambda^\cdot TX)^{\otimes g})$	Verlinde spaces
M^3	RW invariants (finite type)	Reshetikhin-Turaev invariants
Topological boundary conditions	KRS(X) ??	None, usually.

Remark The Cobordism Hypothesis suggests that reasonable (3D) TQFTs T should be "generated by" an object $\mathfrak{X}$ in a symmetric monoidal 3-category.

Eg $Z_{Drinfeld}(\mathfrak{X}) = End(Id_{\mathfrak{X}})$ should be $T(S^1)$.

If lucky: $\mathfrak{X}$ is the 2-category of ∂ conditions for T.

For RT theories that fails: one can define such an $\mathfrak{X}$, but $\nexists$ topological ∂ conditions.

However, the proposed KRS 2-category has so many objects that it could be a generator for $RW(X)$.

KRS proposal

Objects Holomorphic Lagrangians $L \subset X$
with (flat?) bundles $\mathcal{F}$ of (Calabi-Yau?) categories
[2D TQFTs for oriented surfaces]

1-Morphisms $\mathrm{Hom}((L,\mathcal{F});(L',\mathcal{F}')) =$

Write L' near L as the graph of dW, $W: L \to \mathbb{C}$

$= \mathrm{Hom}_{\mathcal{O}_L\text{-mod Cat}}(\mathcal{F}; (\mathcal{F}', W))$.

$MF(L;W)$
if $\mathcal{F} = \mathcal{O}_L$-mod
$\mathcal{F}' = \mathcal{O}_L'$-mod

[Superpotential

[deformation correction if L is not Stein]

Further specification

Localized near $L \subset X$,

$KRS(X) \equiv \mathcal{O}_L$- linear sheaves of DG categories on L.

One key point Grading is by $\mathbb{Z}/2$.

Status: No complete execution to date
Known for $X = T^*Y$ ($\mathbb{Z}$ graded) [Arinkin, others]
Partial def: X = integrable system [–]
Follows: proposal to complete def.

What is KRS(X) and What it ain't

It helps to take the elevator two floors down.
(One floor down is not interesting)

2-categories ⟶ (DG) vector spaces
1-categories ⟶ complex (real) numbers
0-categories ⟶ turtles (T/F).

We can even do differential calculus:

Value	Bundle	Connection	Curvature	Bianchi
Number	Function f	$\emptyset$	df	$d^2f = 0$
Complex	DG bundle $V^\bullet$	$\nabla \oplus \Omega \otimes \mathrm{End}^1$	$\Omega^2 \otimes \mathrm{End}^0 \oplus \Omega^1 \otimes \mathrm{End}^1 \oplus \boxed{\Omega^0 \otimes \mathrm{End}^2}$	✓ deformation class; obstructs sections
Cats	Bundle of DG Cats	$\nabla \oplus \Omega^2 \otimes HCH^0 \oplus \Omega^1 \otimes HCH^1 \oplus \Omega^0 \otimes HCH^2$	$\Omega^3 \otimes HCH^0 \oplus \Omega^2 \otimes HCH^1 \oplus \boxed{\Omega^1 \otimes HCH^2} \oplus \Omega^0 \otimes HCH^3$	Kodaira-Spencer class
2 Cats	Bundles of 2 Cats		$\vdots$ $\Omega^2 \otimes HCH^2$ $\vdots$	eg: sympl. form of degree 2

<u>Eg</u> Connection on $\mathcal{O}_X$-modules from superpotential $W: X \to \mathbb{C}$
Horizontal sections are W-curved complexes.
[Flatness along X is only in antiholomorphic direction]

What KRS is not

The Braiding turns (DG) $\mathcal{O}_X$ into an E_3 algebra $A(X)$
These algebras live in a symmetric monoidal 4-category
Define a (topless) 4D TQFT

Not what we want!

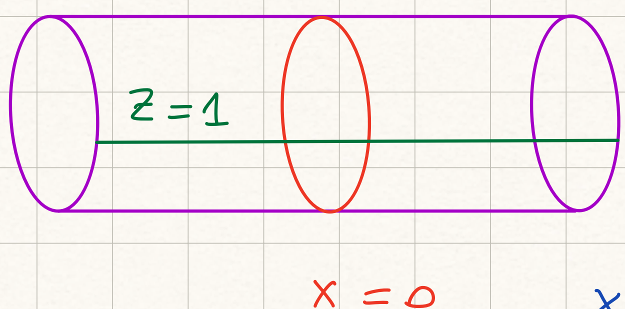
2 floors down: Symplectic form turns $\mathcal{O}(X)$ into an E_1-algebra (LeComte-Wilde, Fedosov) A_X

This defines a (topless) 2D TQFT.
Not the analogue of RW-theory!

Wanted instead "Space of states" representation of A_X

Ideally: sheaf $\mathcal{H}$ over X, $\mathrm{End}(\mathcal{H}) = A_X$ locally.

Let's build it: $X = T^*S^1 = S^1 \times \mathbb{R} = \mathrm{Spec}\, \mathbb{C}[z^{\pm}, x]$.
NonComm. deformation: $z x z^{-1} = x + h$.



Let $\mathcal{H}_0 = A/(x)$

$\cong \mathbb{C}[z^{\pm}]\,\Omega$ as vector space

$x z^n \Omega = z^n (x - nh)\Omega = -nh z^n \Omega$

Spectrum of x: $\mathbb{Z}h$ as expected.

Now try $\mathcal{H}_1^\perp = A/(z-1) \cong \mathbb{C}[x]$ as vector space

Not isomorphic !!
(e.g. x has continuous spectrum)

Because we made a mistake: choosing a Lagrangian

As we vary x, the spaces $\mathcal{H}_a := A \otimes \mathcal{L}/(x-a)$
carry a flat connection. Need global sections.

Base $\mathbb{R}$ is contractible so $\cong \mathcal{H}_0$.

The $\mathcal{H}_b^\perp := A \otimes \mathcal{L}_h/(z-b)$ also carry a flat connection

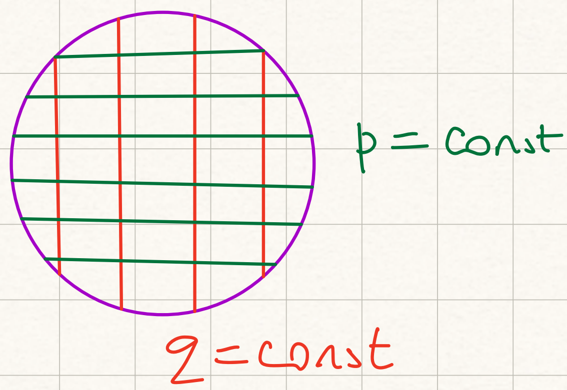
Monodromy $=$ $e^{2\pi i x/h}$ multiplication

Cohomology $=$ $\mathbb{C}[x]/(e^{2\pi i x/h}-1)$ isomorphic to $\mathcal{H}_0$!

This is the isomorphism between the
Bohr-Sommerfeld quantization of T^*S^1
for the two transversal Lagrangian polarizations:

Both spaces are cohomologies of sheaves of sections,
horizontal under the polarizations.

We could conclude independence of polarization if those sheaves were isomorphic:



The map identifying (sections of the line bundle) that depend only on p versus those depending only on q:
FOURIER TRANSFORM

Big Problem: The Fourier transform is not local.

$\Longrightarrow$ No universal proof of independence of polarization.

Big Break: The categorified Fourier transform IS local

1-step categorification: Koszul duality

$$\mathrm{Sym}(V)\text{-mod} \Longleftrightarrow \Lambda V^\vee[-1]\text{-mod}$$
(localized at 0)

2-step categorification:

$$\mathrm{Sym}(V)\text{-mod Cat} \Leftrightarrow \mathrm{Sym}\, V^\vee[-2]\text{-mod cat}$$
(localized at 0)

Should Independence of polarization for KRS 2-category result in

Summary

- Symplectic form ω on X $\longrightarrow$ Bundle of (DG) 2-Cats with connection w/ curvature $\omega \in \Omega^2_X \otimes HH^0$
- For vectn fields V, W (ω should have deg 2!)

alternative $[\nabla_V, \nabla_W] - \nabla_{[V,W]} = \omega(V,W)$

superpotential: deforms $D^b(\mathcal{O}_X)$, the Identity of $(\mathcal{O}_X, \otimes)$-mod

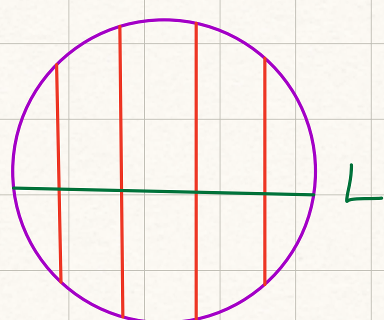
- This can be trivialized along leaves of a Lagrangian foliation

Key Conjecture:
The sheaf of sections that are horizontal along leaves of a Lagrangian foliation is independent of the foliation

Idea of proof: follow Fedosov: build formal germs of the 2-category and take horizontal sections.
[Consistency check still needed]

Remark The stalk at a point of an object in KRS makes no sense as a category (anymore than the numerical value of a section of a line bundle).

To read it one needs a Lagrangian germ L and a cross-foliation (local $\cong$ with T^*L)



The answer depends on this extra structure!
But the sheaf of sections does not.

Analogy Horizontal sections of the quantum line bundle become functions on L
Different L' gives a different function
(shift by φ if $L' =$ graph of $d\varphi$)

Facts (expected)

- The (set-theoretic) support of an object is well defined
- Objects scheme theoretically supported on L have a flat connection along L.
- Interesting families of categories on L will spread out over a neighborhood of L
(finite, if their curvature = KS class is nilpotent)
- Specifying an object in KRS from I requires the cross-foliation (possibly to finite order)
- Objects are equivariant for Hamiltonian flows that preserve their support.

This had better stand on something firm

2 floors down   Tropical geometric Quantization

Fourier transform $\rightsquigarrow$ Legendre transform: Local

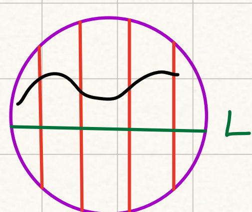
$\Rightarrow$ Have well defined sheaf of tropical states with action of Hamiltonian flows:

Lagrangians with horizontal, unit norm sections of the quantum line bundle $\mathcal{L} \longrightarrow X$

Maslov: "tropicalization = dequantization"

Usual	Tropical
$(\mathbb{R}, +, \times)$	$(\mathbb{R}\cup\{-\infty\}, \max, +)$
Sections of $\mathcal{L}$	$i\log$ of unit norm sections [$\mathbb{Z}$-ambiguity: right thing!]
$\int f(x)\,dx$	$\sup(f)$
$\int f(x)e^{ixy}\,dx$ Fourier	$y \mapsto \sup_x (f(x) - xy)$ Legendre
Quantum mech. Path Int.	Least action principle
$\int_{\text{paths}} \exp\left(i\int_\gamma S/\hbar\right) \mathcal{D}\gamma$	$\inf_\gamma(S)$

Function f on L



Graph of df (where e^{if} is constant)

Change of polarization $p \leftrightarrow q$ Legendre transform

I'd like to say that

The KRS 2-category is homological algebra placed on top of Tropical quantization

I could not quite frame this into a proof, but

Tropical Quantization guides the calculations in KRS.

<u>Example</u> Even categories with nilpotent support near L have a tropical analogue as infinitesimal displacements of L

The distributional sections of $\mathcal{L}$ supported in the n^{th} 'hood of L which are invariant under Hamiltonians form an $(n+1)$-dimensional space $(\dim X = 2)$

<u>Eg</u> $n = 0$: constant sections
$n = 1$: $\langle \delta_0(p),\ q\cdot\delta_0(p) - h\,\delta_0'(p) \rangle$

In favorable cases one can decategorify KRS objects $\mathcal{C}$
They live over their $HH^*(\mathcal{C})$ (internal to KRS)
For Calabi-Yau categories $HH^* \cong HH_*$, the cyclic action on the latter is trivialized.
Turning on the circle action converts X to its def. quant. and $HH^*(\mathcal{C})$ to a module over it

Solutions $\longleftrightarrow$ constant sections
(abstract J-function)

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