Character Theory for topological G-actions on categories

Refs: MSRI lecture 2009
Austin "Perspectives in Geometry" Lecture 2011
Simons Center 2012
ICM 2014
MSRI Lecture 2018

Key anticedents: Olive–Montonen duality (~1977)
Kapustin–Rozanski–Saulina
Kapustin – Gaiotto – Witten "Langlands duality"

Key Remark (Rigidity of topological actions on categories)
Group actions on vector spaces sometimes have no invariant vectors. $\Rightarrow$ possibility of projectors.
Group actions on categories always have invariant categories
$\Rightarrow$ no chance of projectors

1st approximation

Topological G action on $\mathcal{C}$ $\longleftrightarrow$ ΩG-map to $HH^*\mathcal{C}$
(G connected)
look like group actions on vector spaces!

$H_*\Omega G \xrightarrow[(E_2)]{} HH^*(\mathcal{C})$

Executive summary I:

Doubly categorified Fourier transform

1. Abelian case $T = \mathfrak{t}/\Pi$, $T^\vee = \mathfrak{t}^\vee/\Pi^\vee$.

	F.T.		Kernels
Functs on Π	$\longleftrightarrow$	Functs. on $T^\vee$	$\Pi \times T^\vee \to S^1$
$Coh(B\Pi)$	$\longleftrightarrow$	$Coh(T^\vee_{\mathbb{C}})$	$B\Pi \times T^\vee_{\mathbb{C}} \to BGL(1)$
$ShCat(B^2\Pi)$	$\longleftrightarrow$	$ShCat(T^\vee_{\mathbb{C}})$	$B^2\Pi \times T^\times_{\mathbb{C}} \to B^2GL(1)$

$B^2\Pi = B(T/\mathfrak{t})$. So, categories on $B^2\Pi$ = cats. with T-action & trivialized $\mathfrak{t}$-action.

$\equiv$ topological T-action.

Example Topological T-actions on Vect $\Longleftrightarrow$ points in $T^\times_{\mathbb{C}}$

$H^2(BT, \mathbb{C}^\times)$

Closely related to geom. rep theory $\mathcal{D}(G)$-actions

$dR(G) := G/\hat{G}$

For tori distinction just from exponentiation (flat connection vs monodromy)

Nonabelian distinction substantial [Whittaker theory]

Simplified statement Cats with $\mathcal{D}({}_N\backslash G_{\mathbb{C}}/{}_N)_{\chi,\chi}$ - action

$\Updownarrow$

Categories over ${}_{N^\vee}\backslash G^\vee_{\mathbb{C}}/N^\vee$ w/ (χ,χ) twist

$\chi: N \to \mathbb{C}^\times$ regular (Whittaker) character

Eg Fukaya cats of flag varieties $\Longleftrightarrow$ points in ${}_{N^\vee}\backslash G^\vee_{\mathbb{C}}/N^\vee$

Executive Summary II

Real story requires curved categories and $\mathbb{Z}/2$ grading.

The (asymptotic) solutions of the QDE are (Maurer-Cartan) curvings of the respective categories. They are in Legendre duality.

Expect Fourier duality between the genuine solutions (similar to a theorem of Sabbah for $\mathcal{D}$-modules) but one side is periodicized. ($\Rightarrow$ difference equation)

Genuine Solution (O-P; Iritani) $= J \cdot \Gamma$
("Gromov-Witten Γ-function")

Stirling asymptotics $\tau(\log \tau - 1)$ control the deformation

Space: $T^*T_{\mathbb{C}}^{\vee} \cong T_{\mathbb{C}}^{\vee} \times \mathfrak{t}$ (symplectic quantum)

note: $\mathbb{C}[\mathfrak{t}] = H^*(BT;\mathbb{C})$ so "to leading order"

categories with topological T-action
$\longleftrightarrow$ local systems of categories over BT
$\longleftrightarrow$ $\mathbb{C}[\mathfrak{t}]$-module categories
$\longleftrightarrow$ (Sheaves of $\mathcal{O}$-linear) categories over $\mathfrak{t}$.

(NC case: Toda integrable system)

Background [KRS Idea]

There is a problem with the "usual picture" of matrix factorization categories for $X \xrightarrow{W} \mathbb{C}$ holomorphic

Example 1 $Y \subset X$ submanifold.
Physicists tell us that $MF(X;W)|_Y = MF(Y; W|_Y)$.
But if $\mathrm{Crit}(W;X) \cap Y = \emptyset$, the answer should be zero.

Example 2 $MF(X;W)$ is an $\mathcal{O}_X$-module category
In fact $HH^*_{\mathbb{C}}(MF) = (\mathcal{O}_X \otimes \Lambda^\bullet T_X, \iota(dW))$ lives over $\mathrm{Crit}(W)$

But $HH^*_{\mathcal{O}_X}(MF) = \{\mathrm{Hom}_{\mathcal{O}\otimes_{\mathcal{O}}\mathcal{O}}(\mathcal{O}_X;\mathcal{O}_X); \iota(dW)\} \cong \mathcal{O}_X$
lives everywhere!

KRS proposal: They do not localize: they microlocalize.
$MF(X,W) = \mathrm{Hom}_{KRS}(\mathcal{O}_X; MF_W)$ in T^*M.
$\hookrightarrow$ lives on $\Gamma(dW)$.

Example 1 $MF(X;W)$ is $\mathrm{Coh}(\Gamma(dW) \subset T^*X)$
$MF(X;W)|_Y = \mathrm{Coh}(\Gamma(dW|_Y) \subset T^*Y)$ on $\Gamma(dW) \cap T^*_Y X$
from Lagrangian correspondence $T^*Y \leftrightarrow T^*X$ induced by $Y \hookrightarrow X$.

Precise KRS proposal

a 2-category associated to an algebraic symplectic mfld whose objects should include sheaves of (dualizable) $\mathcal{O}_L$-linear categories supported on smooth Lagrangians L
Locally Near smooth L: exactly the 2-cat of $\mathcal{O}_L$-linear categories.

(ICM) Conjecture The KRS 2-category of the BFM space for $G^\vee$ (= Toda integrable system) is the 2-category of G-gaugeable 2D A-models.

Eg $G = T \Rightarrow BFM(G^\vee) = T^* T^\vee_{\mathbb{C}}$.

$\Rightarrow$ "Fourier transform conjecture"

Remark $BFM(G^\vee)$ got re-baptized as "the Coulomb branch for 3-dim $N=4$ pure SUSY gauge theory

And there was much rejoicing

Conjectural part: existence of ambient 2-category.

Much of the rest - true, but lacking a home.

Enough of the KRS proposal can be implemented to give a "home" to the gauge theory results.

(not today's topic)

Two constructions of G-gauged theories

2d TQFTs: generated by categories

A-model group actions: G acts via its topology

$G \hookrightarrow U\mathfrak{g}$-mod by conj

Two interpretations:

- Infinitesimally trivialized actions:
  2-group
  G-action is factored through $G/\hat{G}$. $(\hat{G} \longrightarrow G)$
  Useful in geometric representation theory, less so here
  But DOES give rise to one of the models of equivariantized categories. (Curved Cartan Complex).
- acts trivially
  Locally trivialized actions: $\Omega G \dashrightarrow [PG \longrightarrow G]$
  A bit too homotopical as stated $(G \rightsquigarrow B\Omega G)$
  but leads to a correct picture.
- Rigid Lie model: $\mathfrak{g} \rightsquigarrow$ Lie type $(\pi.G) \otimes \mathbb{C}$
  (complex homotopy type; but leave π_1 as is)

Example (the torus) $T = \mathfrak{t}/\pi$

Topological T-action $=$ $B\pi$-action $=$
(E_2) homomorphism $\mathbb{C}\langle\pi\rangle \longrightarrow HH^*(\mathcal{C})$
$\Longrightarrow$ $\mathcal{O}(T^\vee_{\mathbb{C}})$-module structure on $\mathcal{C}$.

$\Rightarrow$ Fourier decomposition!
Irreducible categorical reps $\longleftrightarrow$ points of $T^\vee$.

The Curved Cartan Complex

Advantages: Explicit.
Contains the "equivariantized category"

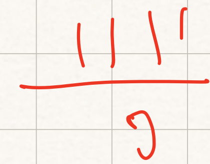
Drawbacks: Differential formulation.
Not so well suited to Fukaya categories
Does not distinguish between $G/\hat{G}$
& topological actions
Algebraic structure of CH^*_G not apparent

Recall the Cartan model for equivariant cohomology

$$\Omega_G^\cdot(X) := \left[\Omega^\cdot(X)\otimes \mathrm{Sym}^\cdot\mathfrak{g}^*\right]^G, \quad \overbrace{d + \Sigma^a \iota(X_a)}^{d_C}$$

$(d_C^2 \neq 0$ without invariance$)$

$G \ltimes$



Promote to a curved algebra

$$G \ltimes \left[\Omega^\cdot(X)\otimes \mathrm{Sym}\,\mathfrak{g}^*\right], \quad d_C, \quad W = \Sigma_a(\delta_1)\otimes 1 \otimes \Sigma^a$$

delta-function on G — fun on $\mathfrak{g}$

Prop. This is a curved algebra, $d_C^2 = [W, \cdot]$.

Modules form a symmetric monoidal category.
Unit object: $\mathrm{Sym}\,\mathfrak{g}^*$
2-cat of module categories localized at $0 \in \mathfrak{g}$:
$\Longleftrightarrow$ 2-cat of $\mathcal{D}(G)$ linear categories.

First definition Categories with (infinitesimal) topological G action = module category over above Tensor category

Invariant category = Hom with unit category.

Eg

$$\begin{tikzcd} G \supset \mathfrak{g} \arrow[r, "L"] & HZ^1(\mathcal{C}) \\ (-1)\,\mathfrak{g} \arrow[u, "\sim"] \arrow[r, "\varepsilon"] & HCH^0(\mathcal{C}) \arrow[u, "\Delta"'] \end{tikzcd}$$

"Lie action"

"interior action"

Requirement: L_∞ morphism $G-$

Eg: ε any map to $HH^0(C)$: deformation over $\mathfrak{g}$.

Remark the deformation family over G is the division by the 'odd' part $\hat{G}$ of $G/\hat{G}$.

If $L = 0$ then ε is a deformation class.

In general it's a deformation class for the G-invariant category.

(In topological case, can easily recover original category from invariants:

locally, G-invariants $\cong \underset{\mathrm{Vect}}{\otimes} \mathrm{Rep}(G)$.

Example For GLSM in V, category is

$(\mathrm{Sym}\,\mathfrak{g}^*\text{-Mod},\ W = \mathrm{Tr}_V[\Sigma(\log\Sigma - 1)])$.

The Toda realization

Advantage: Topological (not differential) construction
Exhibits Fourier transform property

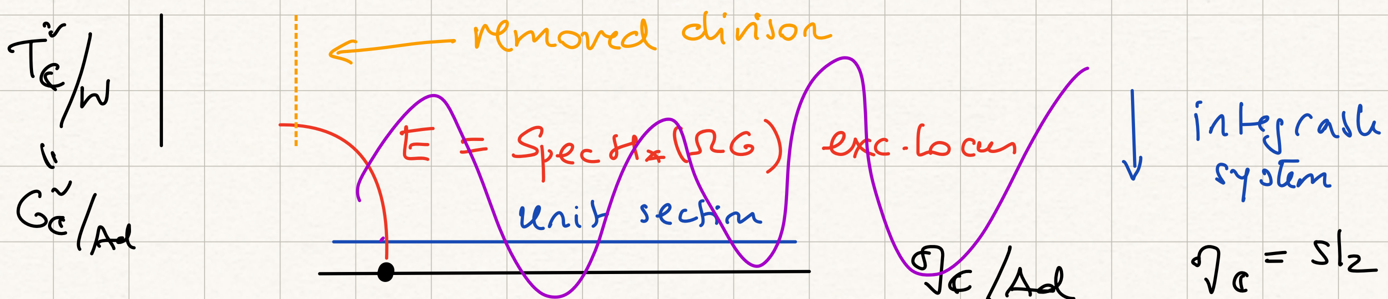
Disadvantage: Constructing "equivariantized" category requires solving the ODE.

(Bezrukavnikov, Mirkovic, Finkelberg, Bielawski, –)

Toda space

- Spec $H^G_*(\Omega G)$ ($\ni QK^*_G(X)$ lives there)
- $N_\chi \backslash\!\backslash T^*G^\vee /\!/_\chi N$ Whittaker reduction of $G^\vee$ — alg. symplectic mfld

  (points in $N\backslash\!\backslash G^\vee /\!/ N$ $\leftarrow$ flag varieties of G)
- $T^*T^\vee_{\mathbb{C}}/W$ resolved (explicitly) & affinized by removing some divisors



Theorem The support of $QH^*_G(X)$ is Lagrangian. Moreover, it defines a difference and differential equation (over $\mathfrak{g}_{\mathbb{C}}/Ad$) valued in $QH^*(X)$.

Pf Turn on the circle action on $H^G_*(\Omega G)$ and QH^*.

Def The equivariantized $\mathcal{F}(X)$ is obtained from $\mathcal{F}(X)$ by the solution to the diff equation (viewed as KS deformation class)

Def The "carefully defined" invariant category $\mathcal{F}(X)^G$ is the global section of the deformation family.

Eg Base X, $W : X \to \mathbb{C}$:

$MF(X;W) =$ global section of deformation family of the category Vect parametrized by X, with KS deformation class dW.

Remark McKay correspondence reduces Coherent calculus in Toda to that in $T^*T_G^\vee/W$ provided we do not meet the missing divisors. This happens for QM_G^τ (compact)

Remark on ${}_{N}\backslash G^\vee / N^\vee$ $\quad (\chi, \chi) \xrightarrow{\deg 2} H^1(; \hat{\mathcal{O}})$

$d\log \to$ vertical shift def of $T^*({}_{N}\backslash G^\vee/N)$

$T^*({}_{N}\backslash G^\vee/N) \rightsquigarrow$ Toda space

$H^1(X;\mathcal{O})$

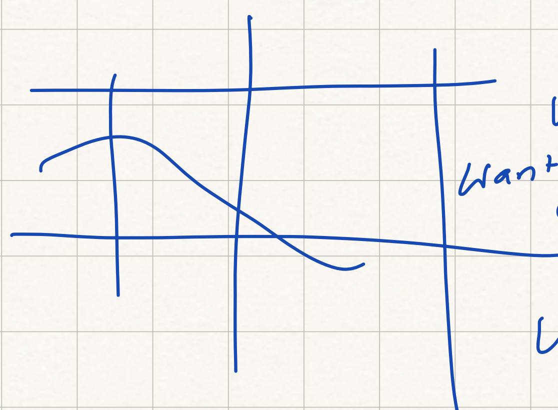
$H^3(X;\mathcal{O}) \subset E_2 HH^3(X)$ - defs of 2-cat of $\mathcal{O}$-module categories

CCC for T, $\mathfrak{t}$

$\mathrm{Spec}(G \ltimes \mathrm{Sym}\,\mathfrak{t}^*, W)$;

$\coprod \mathfrak{t}_\lambda$, $W_\lambda = \xi \mapsto \lambda(\xi)$

$\lambda \in \mathrm{Hom}(T; S^1)$



other cat with W':

without group want $dW' = 0$

with group $dW' = \lambda$

arc λ

Remark There is a kernel for this Fourier transform: Multiplicativity (in T) of the Poincaré gerbe on $T^\vee_{\mathbb{C}} \times B^2\pi$ (w/ flat connection along $T_{\mathbb{C}}$) leads to a bundle of $\mathbb{C}[T^\vee_{\mathbb{C}}] - \mathbb{C}[T^\vee_{\mathbb{C}}]$ bimodules over $T^\vee_{\mathbb{C}} \times B(T_{\mathbb{C}}/\mathfrak{t}_{\mathbb{C}})$.

Or so it would be if the world was nice.

Problem Physics is $\mathbb{Z}/2$ graded not $\mathbb{Z}$-graded. (Eg. QH^*)