Quantization commutes with Reduction AGAIN

* Recent proof (w. Dan Pomerleano) of a long-standing conjecture implicit in the early calculations of GW theory

gauged quantum cohomology = QH^* of symplectic reduction

* True for X compact monotone, at anticanonical reduction if G acts freely; multiplicative correction for finite stabilizers

* Additive correction for non-monotone case or reduction. [Pretty solid conjectures w/ strategy of proof]

* Proof follows outline Floer argument [IAS, 2016, –] with recent additions & checks. Converges with methods of Varolgunes.

* "Twisted sector" construction: TQFT origin, math model: Freed-Hopkins -- construction of Verlinde TQFT and GLSM generalization (–, Woodward)

* Planned as a key application of program for gauging 2D TQFTs by compact groups

* Consequences for categorical result

* 4th theorem in $[Q, R] = 0$ series.

Quantum mechanics — 1 dim QFT

(B) B-model result: Guillemin & Sternberg
(based on Mumford, Kempf–Ness–Hesselink)

(X,ω) compact symplectic
$(\mathcal{L},\nabla)$ prequantum line bundle $\Big\}$ G (cpt connected) acts ($\Rightarrow$ Hamiltonian)

choose Kähler polarization

quantization

reduction (classical) Marsden–Weinstein–Meyer

$(X/\!/G, \mathcal{L}/\!/G) = \mu^{-1}(0)/G$

$H^0(X;\mathcal{L})$ for a positive cx Polarization

quantization

reduction (quantum)

$$H^0(X;\mathcal{L})^G \xrightarrow{\quad} H^0(X/\!/G;\mathcal{L}/\!/G)$$

natural!

Theorem (G–S) The natural map is an isomorphism.

Proof (not the original)
$H^*(X;\mathcal{L})^G$ and $H^*(X/\!/G;\mathcal{L}/\!/G)$ differ by $H^*_{US}(X;\mathcal{L})^G$
US = unstable locus = union of unstable Morse strata for $\|\mu\|^2$

Lemma $H^*_S(X;\mathcal{L})^G = 0$ for any unstable stratum
because there's a 1-parameter subgroup killing it.

* This shows how the result can fail w/o positivity (eg $H^*_S(X;\mathcal{E})^G$ may be $\neq 0$ for other $\mathcal{E}$)
* Idea will carry over to 2D symplectic case.

Quantum Mechanics – 1 dim QFT

(A) A-model result: Kirwan surjectivity

Replace "holomorphic" with "homotopic"
Coherent cohomology $\longrightarrow$ rational cohomology

$$\begin{tikzcd}
(X,\omega) \arrow[rr, dotted, "\text{classical reduction}"] \arrow[d, dotted, "\text{quant.}"'] & & (X/\!/G;\omega) \arrow[d, dotted, "\text{quantization}"] \\
H^*(X;\mathbb{Q}) \arrow[r, dotted, "\text{quantum}", "\text{reduction}"'] & H^*_G(X;\mathbb{Q}) \arrow[r, "\text{natural}"] & H^*(X/\!/G;\mathbb{Q})
\end{tikzcd}$$

Theorem ("Kirwan surjectivity") The natural map is onto, and is the top part of a filtration of $H^*_G(X;\mathbb{Q})$ whose Gr comprises the $H^*_{S;G}(X;\mathbb{Q})$.

Reflective Pause Why the difference in statements?

B-model: Concerns linear representations of compact groups
A-model: Concerns topological representations.

(B) is controlled by a rigid character theory
(and is moreover semi-simple)
(A) is entirely derived.

So it is completely reasonable that clever constructions lead to projections onto the desired part in (B) but only to filtrations in (A).

2D QFT: Surprise, the stories are reversed

2D TQFTs ought to be generated by linear categories.
(A): some version of the Fukaya category
(B): $D^b Coh(X)$

(B) Daniel Halpern-Leistner; Ballard-Favero-Katzarkov

Thm $D^b Coh(X)^G$ has a semi-orthogonal decomposition in which one piece is D^b(stacky $X/\!\!/G$).

Remark Additional choices are needed and this 'bug' is really a feature; $\Rightarrow$ non-obvious derived equivalences.

(A) Expect in the positive (monotone) smooth case

$$\mathcal{F}(X)^G \equiv \mathcal{F}(X/\!\!/G)$$

↑ carefully defined! (tomorrow)

We think we can prove this with our methods, but
(a) this does not reconstruct the GW TQFT
(b) the logical implications now go $QH^* \rightsquigarrow \mathcal{F}(X)$ via the closed-open map.

We prove instead the theorem for the closed sector.

Theorem (Pomerleano; –)

X compact, monotone, $\mathcal{L} = K^{-1}$, G action free on $\mu^{-1}(0)$:

$$QH^*(X/\!\!/G) \cong QH^*_{LG}(X) \quad \text{as algebras.}$$

Amplification: $QH^*(X/\!\!/G) = QH^*_G(X) \otimes H^*(BG)$
$H^G_*(\Omega G)$ Toda space

and the tensor product is strict (no $\mathrm{Tor}_{>0}$).

Addenda (1) If G acts locally freely (orbifold case) the additive isomorphism holds but we expect a deformation of the multiplication.

(2) Away from the monotone reduction, there is "trapped cohomology" in X: the map "$\leftarrow$" is surjective.

(3) Even more so when X is not monotone.

Remark We have a pretty good idea about $\uparrow$ and can prove it for S^1.

General case & trapped cohomology: analogous to $H^*_S(X; \mathcal{E})^G$

Shift operators replace the Hilbert-Mumford 1-parameter subgroups.

LG equivariant QH^*

Morally, $QH^*(X) = H^*(LX)$ (Morse theory)

If G acts on X, LG acts on LX

$\Rightarrow$ should have $QH^*_{LG}(X)$

Method: Monodromy representation.

For $g \in G$, $\rightsquigarrow HCF^*(X;g)$

- (derived) local system on G
- multiplicative
- G-equivariant for conjugation

<u>Prop</u> (General nonsense)

Fact: $QH^*_{LG}(X) : H^G_*(G; \mathcal{H}F^*)$

(1) The monodromy representation gives an ΩG action on $HCF^*(X)$.

(2) The monodromy –"– .. $\Omega G \rtimes_{Ad} G$ action on $CF^*_G(X)$.

(3) $CH^G_*(G; \mathcal{H}CF^*(X))$ is an E_2 algebra

(4) $$\cong \overset{E_2}{CF^*_G(X)} \overset{L}{\otimes}_{\underset{(E_3\ \text{alg})}{CH^G_*(\Omega G)}} \overset{E_2}{CH^*(BG)}$$ by the monodromy rep.

(5) There is even a circle action for the unbased statement.

$(\Omega G)/_{AdG} \cong G \backslash LG / G$ circle action

This defines the meaning of the ingredients.

Proof of the Theorem

Ideal proof: have a principal LG fibration

$$LG \hookrightarrow X \twoheadrightarrow X/\!\!/G.$$

Obviously false, but true in Floer theory!

The $|\mu|^2$ Hamiltonian

Define the Floer complex by the Hamiltonian $\frac{1}{2}K|\mu|^2$, $K\to\infty$

Short story In the monotone case, all orbits eventually continue into a neighborhood N of $\mu^{-1}(0)$.
That is a principal T^*G- bundle over $X/\!\!/G$. symplectic normal form thm.
An energy estimate shows that

the portion of the Floer complex in N has a decreasing filtration with components related by q-powers whose Gr is $\cong H^*(X/\!\!/G; SH^*(T^*G))$
$\hookrightarrow H_*(LG)$.

Continuation to N follows by an easy argument in the monotone case: the monotone index of orbits
$=$ (Floer degree) $- 2\cdot$action
can be estimated to $\geq K\mu^2$.
But action is essentially ≥ 0 in fixed degree $\Rightarrow |\mu|\to 0$.

LG-equivariant

Longer story: The part of the Floer complex inside N, as $K \to \infty$, is always additively quisomorphic to Floer complex of the base $X/\!/G$ (orbifold)

This is because the complex can be fibered over G, and its terms are now open geodesics in G.

That space is $G \times \mathfrak{g}$; G-equivariance kills G leaving a contractible factor.

This construction is also strictly compatible with the Lagrangian correspondence $\mu^{-1}(0) \subset X \times X/\!/G$ showing that the isomorphism is induced by Σ.

This argument does not use monotonicity (but does not recognize product structures).

Away from monotonicity

The monotone index argument fails, and we must examine the continuation map.

Possible snags: critical points of $|\mu|^2$.
Leading order calculation: Cohomology gets trapped wherever the weight of the Hilbert-Mumford subgroups on K^{-1} of a fixed point set is negative.
$\Longrightarrow$ Equivariant cohomology is trapped there.

Conjectural description in general (non-monotone)

(1) The ideal $J_\infty < QH^*_G(X)$ spanned by Floer orbits whose action $\longrightarrow \infty$ as $k \to \infty$ is trapped, and

$$QH^*(X/\!/G) \underset{\text{multiplicative deformation}}{\overset{\text{additive}}{=\!=}} QH^*_G(X)/J_\infty \otimes \underset{H^G_*(RG)}{H^*(BG)}.$$

[Very high degree of confidence]

(2) For $G = S^1$ when we have no zero-weight on K over any fixed point:

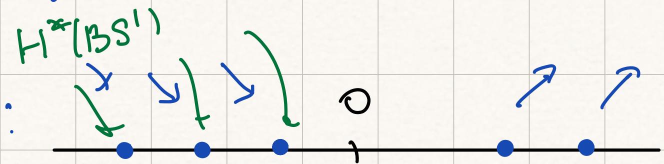
$QH^*_{S^1}(X)$ is free over the shift operators with rank

$$\sum_{F;\, w(F)>0} w(F) \cdot \operatorname{rank} H^*(F)$$

F fixed pt set, $w(F)$ weight on K

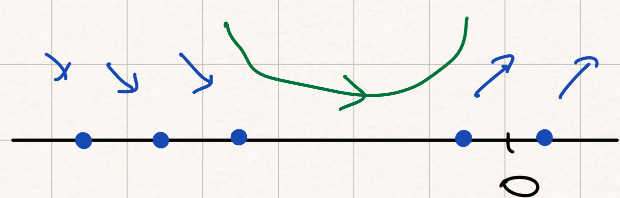
The shift operators extract cohomology at $w(F)>0$ and push it into the bulk at $w(F)<0$

monotone:



all cohomology "passes through" $\mu^{-1}(0)$

monotone shifted



some cohomology misses $\mu^{-1}(0)$

[High degree of confidence]

(3) [Speculative]

The trapped cohomology in $QH^*_G(X)$ is the image of the symplectic cohomologies w/ support (Varolgunes) at the wrong-weight fixed pt. sets.

Remark No easy (semi) classical description ?