Fully local Reshetikhin-Turaev theories

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Abstract

We define a symmetric tensor enhancement $\mathbb{EF}$ with full duals of the 3-category $\mathbb{F}$ of fusion categories in which every Reshetikhin-Turaev theory has a fully local realization. Our $\mathbb{EF}$ is a direct sum of invertible $\mathbb{F}$-modules, indexed by a $\boldsymbol{\mu}_6$-extension of the Witt group W of non-degenerate braided fusion categories. Similarly, we enhance the 3-category $S\mathbb{F}$ of fusion super-categories to a symmetric tensor 3-category $\mathrm{E}S\mathbb{F}$ with full duals, which is a sum of invertible $S\mathbb{F}$-modules indexed by an extension of the super-Witt group SW by the Pontrjagin dual of the stable stem π_3^s. The unit spectrum of $\mathrm{E}S\mathbb{F}$ is the connective cover of the Pontrjagin dual of $\mathbb{S}^{-3}$. We discuss tangential structures and central charges of the resulting TQFTs, and establish the Spin- and SO-invariance properties, in relation to modularity, sphericality, proving some conjectures from [DSPS].

Introduction

(0.1) Background. The 3-dimensional topological quantum field theories (TQFTs) constructed by Reshetikhin-Turaev [RT] and Witten [W] are (symmetric monoidal) functors from a 2-category of compact manifolds of dimensions $1, 2, 3$ to the target symmetric tensor 2-category $\mathbb{L}$ with objects finite, semi-simple $\mathbb{C}$-linear categories (linear over the symmetric monoidal category $\mathbb{V}$ of complex vector spaces), and with functors and natural transformations as 1-and 2-morphisms. On oriented manifolds, the theories are only *projectively* defined. The projective dependence is controlled by a *central charge* $c \in \mathbb{C}$, normalized to align with the Virasoro central charge of (rational) conformal field theories, whose chiral sectors live on the boundary of these TQFTs. Thus, the chiral free fermion conformal theory has $c = 1/2$, and lives on the boundary of the *topological free fermion* theory ψ, an invertible, albeit Spin-dependent TQFT. Restricted to framed manifolds, ψ defines the standard character $\pi_3^s \to \boldsymbol{\mu}_{24}$ of the framed bordism group.

The projective anomaly in the TQFT gluing laws can be resolved by specifying a suitable 3-*dimensional tangential structure* τ, and supplying all manifolds with a τ-structure on a 3-dimensional germ. Certain shifts in the structure multiply the output by c-dependent phases, acting as top-level automorphisms; this also renders c ambiguous by a (τ-dependent) additive shift. This dependence is easy to explain: a change of structure that can be implemented locally, near a single point of the manifold, affects the theory by point operators, which in these theories are scalar. By contrast, changes in Spin structure are not localized at single points, and Spin-dependent versions of RT theories show a more complicated dependence on the structure [B, J].

Several tangential structures may be used. The default is a 3-*framing;* a shift in framing by $1 \in \pi_3\,\mathrm{SO}(3) = \mathbb{Z}$, which can be effected near a single point, transforms the invariant of a closed, framed 3-manifold by a factor[1] of $\exp(2\pi ic/6)$. Less restrictive is an SO^{p_1}-*structure*[2] (oriented p_1-structure), a trivialization of the Pontrjagin cocycle p_1 on oriented manifolds; this was first considered in [BHMV]. A unit shift in p_1-structure changes the top-level invariants by $\exp(2\pi ic/24)$, reflecting the value $p_1 = 4$ on the basic spherical class in $B\mathrm{SO}(3)$. Mediating between these options are Spin^{rp_1}-structures, incorporating trivializations of $(r\cdot)p_1$ on Spin manifolds; we review those in §3, with more detailed computations in Appendix B. One novelty we introduce in §7 is a $\mathbb{C}p_1$-*structure*, which sees a lift the central charge to $\mathbb{C}$.

(0.2) Caution on central charges. The four-fold relation between framing- and p_1-shifts does not persist for general tangential structures. It is broken by the invertible 3-framed theory ν of order two (see §3.9), whose central charge cannot be consistently lifted mod 24: it extends to a Spin^{p_1}-theory with central charge 0, as well as an oriented p_1-version with central charge 12. The problem stems from the co-kernel $\mathbb{Z}/2$ of $J : \pi_3\mathrm{SO}(3) \to \pi_3^s$. (Recall that J surjects from $\pi_3\mathrm{SO}$.) This ν detects the unstable $\mathbb{Z}/2$ summands in several 3-dimensional bordism groups (Table 2 in Appendix B). Because of this, ν is not *reflection-positive* theory in the classification of [FH]. It is therefore not clear whether the central charge ambiguity affects *unitary* TQFTs, and we do not discuss unitary structures in this paper.

(0.3) RT theories from categories. An individual RT theory $\mathscr{T}$ is determined by

(i) a finite, semi-simple category $T := \mathscr{T}(S^1)$, associated to the standard circle,
(ii) the rigid, braided tensor structure on T, defined from the pair-of-pants multiplication,
(iii) the *ribbon* automorphism of the identity of T, defined by the circle rotation action.

The data (i)–(iii) underlie the notion of a *modular tensor category*, which is subject to two constraints:

- *non-degeneracy* of the braiding: the central objects form an additive summand $\mathbb{V} \subset T$,
- *symmetry* of the ribbon, giving a homogeneous quadratic enhancement of the braiding.

Viewing T as an object in a 4-category of braided fusion categories[3] (BFCs), the ribbon trivializes the square of the braiding,[4] which is the structural *Serre automorphism*: see §5. Symmetry ensures that the trivialization has order 2. These data and constraints *very nearly* determine a TQFT $\mathscr{T}$ for oriented manifolds with *signature structure* in dimensions 1/2/3: see for instance Turaev's book [Tu], the detailed account in [BDSV], or our brief discussion in §8. Left over is a sign ambiguity (Remark 0.4 below). The central charge $c(\mathscr{T})$ now couples to $1/8^{th}$ of the signature. A (logarithm of a) Gauss sum, defined from T and its ribbon [M], gives a rational number mod $8\mathbb{Z}$, which agrees with $c(\mathscr{T})$ mod $4\mathbb{Z}$.

0.4 Remark. Our saboteur, coker J, forces an amendment to the old folk belief that the Gauss sum determines $c(\mathscr{T})$ mod $8\mathbb{Z}$; this had already been noted [BK, BDSV]. There are *two* signature-structured TQFTs for oriented manifolds defined by the same modular T, differing by the invertible theory ν. On oriented manifolds with signature structure, the latter has central charge 4 mod $8\mathbb{Z}$. We explain this sign problem in §8, in terms of *half-signature structures*: the choice of sign arises in their promotion to signature structures.

We prefer the setting of p_1-structures, which genuinely localize to points. (Signature and half-signature structures only do so projectively, see §8.) Then, an anomalous TQFT $\mathscr{T}$ has 6 fully local linearizations on SO^{p_1}-manifolds. Their central charges, defined mod 24, agree with the Gauss

[1] Some early literature normalized c incorrectly, by a factor of 2.
[2] This is sometimes called a (w_1, p_1)-structure.
[3] There are several variant 4-categories; for this statement, E_2 objects in linear categories will do, see e.g. [BJS].
[4] Equivalently: in the pair of pants, the square of the braiding is the product of the three boundary Dehn twists.

sum mod 4. The theories are related by powers of an invertible theory $\mathscr{T}_U$ we will meet below, and differ by the characters $\pi_3^s \to \boldsymbol{\mu}_6$. From this perspective, signature structures are an (unsuccessful) attempt to remove the linearization ambiguity by killing the even powers of $\mathscr{T}_U$. However, the odd ambiguity remains.

Spin versions of RT theories can be defined using braided *super-categories* — linear over super-vector spaces — which meet a similar non-degeneracy condition: the central objects form a copy of $S\mathbb{V} \subset T$. There will be 24 choices of framed theories for a given T, differing by powers of the topological free Fermion theory ψ (see §3.9). Other tangential choices lead to different sets of options; for instance, Spin^{p_1}-structures give 48×2 choices.

(0.5) Formulation of the problem. Although 'one step more local' than Atiyah-Segal TQFTs [A] in their extension to corners of co-dimension 2, Reshetikhin-Turaev theories are usually not *fully local* in the sense of the Baez-Dolan-Lurie *Cobordism Hypothesis* [L]: they are not generated by an object $X = \mathscr{T}(pt)$ in an obvious symmetric tensor 3-category. Indeed, for such an X, the endomorphism category of Id_X — the *Drinfeld center* $Z(X)$ — should be braided-tensor isomorphic to T. The obstruction to finding a *fusion category* X with these properties is (essentially by definition) the class $[T]$ in the *Witt group* W of non-degenerate braided fusion categories [DMNO], the quotient of invertible BFCs by Drinfeld centers. If $[T] = 0$, then such an X exists, uniquely up to isomorphism in $\mathbb{F}$; the resulting fully local theory is then a *Turaev-Viro TQFT*. Absent such a fusion category X, the theory $\mathscr{T}$ is more difficult to access.

The question of fully localizing RT theories, especially in relation to Chern-Simons theories, has received some attention; constructions have been proposed in terms of vertex algebras and their modules, or nets of von Neumann algebras, based on their conformal boundary theories [K, He]. A feature of these constructions is their use of *additional analytic input*, and the apparent *absence of additional topological output*. This leads to the suspicion that the TQFT information required to fully localize these theories is entirely contained in their 1/2/3 portion.

(0.6) Results. In this paper, we confirm this suspicion, and enlarge the 3-category $\mathbb{F}$ of fusion categories to a universal *symmetric monoidal* 3-*category* $\mathrm{E}\mathbb{F}$ ("enlarged $\mathbb{F}$"), containing the point generators of Reshetikhin-Turaev theories, with the following properties:

(i) $\mathrm{E}\mathbb{F}$ *has full duals*: all k-morphisms are $(3-k)$-dualizable, $0 \le k \le 3$;

(ii) $\mathrm{E}\mathbb{F} = \bigoplus_{w \in \widetilde{W}} \mathbb{F}_w$, where $\widetilde{W} \to W$ is an extension of the Witt group by $\boldsymbol{\mu}_6$;

(iii) There are no non-zero morphisms $\mathbb{F}_w \to \mathbb{F}_{w'}$ when $w \neq w'$;

(iv) Each $\mathbb{F}_w$ is an invertible module over $\mathbb{F}_1 = \mathbb{F}$; specifically, choosing a representative braided category $T(w)$ for w gives an isomorphism $\mathbb{F}_w \equiv \mathbb{F}_{T(w)}$, the 3-category of fusion categories with central action of $T(w)$ (called *fusion categories over* $T(w)$);

(v) When $\zeta = \exp(2k\pi i/6) \in \boldsymbol{\mu}_6 \subset \widetilde{W}$, we have $\mathbb{F}_\zeta \equiv \mathbb{F}$ as a module, generated by an invertible object $U^{\otimes k}$, unique up to (Morita) isomorphism;

(vi) The units $U^{\otimes k}$ generate the six invertible framed TQFTs valued in $\mathrm{E}\mathbb{F}$, and factor uniquely through the category of oriented manifolds with p_1-structure;

(vii) Every symmetric monoidal 3-category whose looping contains $\mathbb{L}$ and where all 1/2/3 framed TQFTs have point generators that are unique up to isomorphism receives a unique symmetric tensor functor from $\mathrm{E}\mathbb{F}$.

(0.7) Commentary. In (vii), we would like to state that $\mathrm{E}\mathbb{F}$ is universal in promoting Reshetikhin-Turaev theories to fully local, framed theories, but this needs correction: not every non-degenerate BFC is modular, and the modular structure is not unique; so such a category may only receive a functor out of a 'modular part' of $\mathrm{E}\mathbb{F}$.

Most properties are largely forced by the final condition (vii). For instance, whereas (iii) could seem arbitrary, it follows from the main result of [FT1]: no non-zero topological interfaces exist between Witt inequivalent, fully local RT theories. This also counters a traditional supposition, namely that the point generator for a 3D TQFT must be the 2-category of its topolgical boundary theories: none such exist for objects in $\mathrm{E}\mathbb{F} \setminus \mathbb{F}_1$. Property (vi) is specific to *bosonic* theories, with no super-vector spaces in the target. A consequence is that the subgroup $\mathbb{Z}/4 \subset \pi_3^s \cong \mathbb{Z}/24$ is represented *trivially* in a bosonic, invertible 3D theory, and π_3^s must factor through the bordism group $\mathbb{Z}/6$ of p_1-oriented 3-manifolds. (This last group is also $\pi_3(\mathbb{S}/\eta)$, see Remark 1.11.)

(0.8) Super-Results. The same method gives a super-version of this result, relevant for Spin RT theories, considered early on in [BM, B], or more recently [J]: we get an enlargement $\mathrm{E}S\mathbb{F}$ of the 3-category $S\mathbb{F}$ of fusion super-categories by the super-Witt group SW of non-degenerate BFSCs modulo Drinfeld centers from $S\mathbb{F}$. The kernel of the extension $\widetilde{SW} \to SW$ is now the Pontrjagin dual of $\pi_3^s = \mathbb{Z}/24$, and the objects of $\mathrm{E}S\mathbb{F}$ generate the framed 3-dimensional TQFTs.

(0.9) Four-categorical aspects. If we invoke the recent classification of fusion 2-categories by [CF2], we can continue our discussion in one higher dimension. Namely, the new objects $X \in \mathrm{E}\mathbb{F}$ become new 1-*morphisms* in the symmetric tensor 4-category of fusion 2-categories: specifically, isomorphisms of their centers $Z(X)$ with the unit category. This should kill the (super-)Witt group, and produce a fully dualizable symmetric monoidal 4-category of point generators for 4-dimensional TQFTs, where all theories become finite gauge theories with generalized Dijkgraaf-Witten twists. Conjecturally [LKW], these are *all* the fully local, 4-dimensional TQFTs valued in 4-categories (as opposed to $(\infty, 4)$-categories), so the output has the flavor of a universal target for 4-dimensional TQFTs. A version of this conjecture was established in [CF2]. However, the new objects in $\mathrm{E}\mathbb{F}$ give rise to a larger 4-category (see for instance the novel $\mathbb{Z}/3$-gauge theory in Appendix A). Unlike the 3-dimensional case, a precise universality property needs fleshing out. A model for such characterization, and for its generalization to higher dimensions, has been announced by Johnson-Freyd and Reutter. We will return to the 4-dimensional construction in a future paper [FST].

(0.10) Orientations and spherical structures. We also address the question of relaxing the manifold structures on the domain of these (super) Reshetikhin-Turaev TQFTs. *A priori*, they require 3–framings, and their p_1-dependence precludes a clean factorization through Spin or oriented manifolds. However, the topological boundary theories for Turaev-Viro (super) TQFTs limit their p_1-dependence, and force the vanishing of (appropriately reduced) central charges, descending them to Spin and, at times, oriented TQFTs.

For fusion categories F, we confirm the long-assumed relation (partially established in [Tu]) between modular structures on the center with $\mathrm{SO}(3)$-invariance data of F as an object of $\mathbb{F}$. However, not all modular structures are created equal. If they exist, there is a preferred one; it has the property that *all boundary theories are also* SO-*invariant* (Theorem 6). For general modular structures, a sign obstruction to $\mathrm{SO}(3)$-invariance is detected by a certain braided fusion category (6.6), which can be isomorphic to either $\mathbb{Z}/2$-graded $\mathbb{V}$ or $S\mathbb{V}$. The preferred modular structure defines a distinguished spherical structure on F. Other spherical structures arise from central lifts of order 2 of $\mathbf{1} \in F$, and they preserve the SO-invariance of the regular module, the *Dirichlet boundary theory*. More precisely, spherical structures on F correspond to an SO-*invariance data for F together with its regular module* (Theorem 4). This settles some open conjectures of [DSPS], and meshes well with the main theorem of [FT2], which characterized fusion categories as 3-dimensional simple TQFTs equipped with a non-zero boundary theory (the regular module).

(0.11) Related prior work. Our theorems rely strongly on the dualizability properties of fusion and braided fusion categories, established in [DSPS, BJS]. Another key input to our construction is Kevin Walker's [Wa] presentation of Reshetikhin-Turaev theories as *fully local anomalous* TQFTs; see [H] for a modern exposition of that construction. Specifically, a modular tensor category T generates a fully local, *invertible* 4-dimensional TQFT of oriented manifolds. Invertibility follows from that of T in the higher Morita category of BFCs, established in [DMNO]. The *regular module*, the fusion category T over T, provides a fully local boundary theory. This anomalous presentation of RT theories leads to their description as linear theories in dimensions $1/2/3$, due to the vanishing of the oriented bordism groups in those dimensions. We extend this to dimension 0.

We know of an approach at constructing a target 3-category proposed by Kong [K], in terms of boundary vertex algebras; however, even with the detailed work in [KYZZ], the checks needed to obtain a symmetric structure did not seem to be identified.

(0.12) Extended summary. Here is a guide to the subsequent sections of the paper.

1. Section 1 constructs the bosonic enlarged category $\mathbb{EF}$, explaining the role of the symmetric group.
2. Section 2 repeats the construction for fusion super-categories, leading to $\mathbb{E}S\mathbb{F}$.
3. Section 3 reviews the projective Spin-invariance of TQFTs (Theorem 3.2) and the topological central charge $\mu = \exp(2\pi i c/6)$. Tangential structures related to p_1 on Spin manifolds are related to anomalies (Theorem 3).
4. Section 4 proves the $\mathrm{Spin}(3)$ invariance of TQFTs defined from super-categories, and of interfaces between them, confirming a conjecture from [DSPS].
5. Section 5 reviews the removal of Spin structures in the presence of modularity.
6. Section 6 classifies orientation structures on Turaev-Viro theories and characterizes the *canonical* orientation.
7. Section 7 introduces the *complex* p_1-structures, enabling the lift of the central charge to $\mathbb{C}$. We verify its match with the central charge of a boundary CFT.
8. Section 8 introduces *projective* symmetric monoidal structures, and displays their use in fully localizing the *signature structures* of [Tu] and their half-signature versions.

A. This Appendix shows how the new objects in $\mathbb{EF}$ lead to novel 4-dimensional TQFTs, with the example of a 3-dimensional gauge theory.

B. This appendix reviews some relevant bordism groups and maps between them.

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Notation and abbreviations.

- TQFTs = topological quantum field theories
- BF(S)Cs = braided fusion (super-)categories
- $\mathbb{V}$ is the $\mathbb{C}$-linear category of finite-dimensional vector spaces
- $\mathbb{L}$ is the linear 2-category of finite semi-simple $\mathbb{C}$-linear categories
- $\mathbb{V}^{\otimes}$ the same with its tensor structure
- $\mathbb{V}^{br\otimes}_{\mathbb{Z}/2}(\zeta)$ the BFC of $\mathbb{Z}/2$-graded vector spaces with one of the four braided structures, labelled by $\zeta \in \{\pm 1, \pm i\}$
- $\mathbb{F}$ = (symmetric tensor) 3-category of: fusion categories, finite semi-simple bimodule categories, functors and natural transformations
- $\mathbb{B}$ = (symmetric tensor) 4-category of BFCs: a full subcategory of algebras in $\mathbb{F}$, with bimodules in $\mathbb{F}$ as 1-morphisms and compatible higher morphisms[5]
- W = Witt group of invertible BFCs modulo centers
- Prefixed S means 'super' (including super-vector spaces, Clifford algebras, etc.): $S\mathbb{V}$, $S\mathbb{L}$, $S\mathbb{V}^{\otimes}$, $S\mathbb{F}$, SW, $S\mathbb{B}$ (cf. also §2)
- $\mathbb{S}$ is the sphere spectrum, HA the Eilenberg-MacLane spectrum for an Abelian group A
- $\mathbb{I}_{\mathbb{C}^\times}$ is the Pontrjagin dual of $\mathbb{S}$
- $\mathscr{T}_X$ denotes the TQFT defined by an object $X \in \mathrm{E}S\mathbb{F}$ (framed, unless otherwise stated)
- $\mathrm{GL}_1 T$ denotes the (higher) group of invertibles in a tensor (higher) category T
- $\iota_{>k} C$ denotes the (higher) subcategory of C keeping only invertible j-morphisms for $j > k$
- $[x]$ denotes the isomorphism class of an object x in the ambient (higher) category
- For a characteristic class κ of $B\mathrm{Spin}(n), B\mathrm{SO}(n)$, denote by $\mathrm{Spin}^\kappa(n), \mathrm{SO}^\kappa(n)$ the groups classified by the homotopy fibers of κ (as maps to the respective Eilenberg-MacLane space).

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1. Main result: bosonic case

We state and prove the main theorem for *bosonic* TQFTs — those not including super-vector spaces — and follow up with some additional insights on the construction.

Theorem 1. *There exists a symmetric monoidal* 3*-category* $\mathrm{E}\mathbb{F} \equiv \bigoplus_{\widetilde{W}} \mathbb{F}_w$ *satisfying properties (i)-(vii) in §0.6, unique up to isomorphism.*

[5]See [JS] for an iterative construction of higher categories of algebras.

The units in $\mathbb{EF}$ form a cyclic group of order 6; they are characterized by their behavior under symmetry (Remark 1.7). Their reduced central charges are in $2\mathbb{Z} \pmod{6\mathbb{Z}}$, but lift to $4\mathbb{Z} \pmod{24\mathbb{Z}}$ under refinement of structure (see §3). Here is an amusing consequence:

1.1 Corollary. *The framed* 3*-manifold invariants seen by* invertible *bosonic theories are the characters of* $\pi_3^s \cong \mathbb{Z}/24$ *which factor through* $\mathbb{Z}/6$.

We denote by $U \in \mathbb{EF}$ the object defining the fourth power of the standard character of π_3^s. By contrast, the fermionic theories in the next section can represent the full $\mathbb{Z}/24$.

We construct $\mathbb{EF}$ by converting Walker's relative (anomalous) RT theories to absolute ones. Trivializing the $4D$ anomaly theory defined by modular T requires an object $X \in \mathbb{EF}$, together with a Witt equivalence $T \equiv Z(X) := \mathrm{End}(\mathrm{Id}_X)$. If $[T] \neq 0$ in the Witt group, X cannot be a fusion category. Instead, we add the missing pre-centers as direct summands to $\mathbb{F}$, as follows. Recall that for a fusion category F, $Z(F) \equiv F \boxtimes F^\vee$ by a Morita equivalence in $\mathbb{F}$ which matches the second algebra structures on the two sides (the braiding with the contraction). The imagined generators $\{X\}$ of a collection RT theories, whose centers $\{T\}$ represent the non-trivial Witt classes, will now be portrayed by their centers T (viewed as fusion categories over T), whereupon we rig the tensor product so that $T \boxtimes T^{rev}$ becomes isomorphic to T, to match what $X \boxtimes X^\vee$ should produce in $\mathbb{EF}$. We must of course be coherent with respect to the symmetric tensor structure.

Proof. For each $w \in W$, choose a representing non-degenerate braided fusion category $T(w)$, denote by $\mathbb{F}_w$ the category of fusion categories over $T(w)$, and define (provisionally)

$$E'\mathbb{F} := \bigoplus\nolimits_{w \in W} \mathbb{F}_w$$

with only zero morphisms between distinct summands. Each summand is an invertible $\mathbb{F}$-module, because of the invertibility of the categories $T(w)$ in $\mathbb{B}$.

An obvious attempt at placing a symmetric tensor structure on $E'\mathbb{F}$ matches the addition law in the Witt group, and is implemented by multiplications

$$\mathbb{F}_w \boxtimes \mathbb{F}_{w'} \to \mathbb{F}_{ww'}.$$

To execute these operations, we must use isomorphisms $T(w) \boxtimes T(w') \cong T(w+w')$ in $\mathbb{B}$. For instance, if $w' = -w$, and we have opted for the reverse-braided category $T(w)^{rev}$ to represent $(-w)$, the tensoring operation must be done *over* $T(w)^{rev}$. This is how the object $X = T(w)$ satisfies $X \boxtimes X^\vee \equiv T(w)$, by means of an equivalence as *algebra objects in* $\mathbb{F}$. This attempt meets obstructions and ambiguities from the incoherent choice of representatives $T(w)$.

The obstruction problem is controlled by cocycles valued in the (higher) groups of automorphisms of the invertible $\mathbb{F}$-module categories $\mathbb{F}_w$. These all agree with the group $B^3\mathbb{C}^\times$ of units of $\mathbb{F}$. More precisely, the Witt group W has a natural categorification to the 4-group $\mathrm{GL}_1(\mathbb{B})$ of invertible BFCs, and we have

$$\pi_0\,\mathrm{GL}_1(\mathbb{B}) = W, \qquad \pi_4\,\mathrm{GL}_1(\mathbb{B}) = \mathbb{C}^\times,$$

the other groups being zero because of the absence of non-trivial invertible objects in $\mathbb{V}$, $\mathbb{L}$ and $\mathbb{F}$.[6] The symmetric tensor structure on BFCs defines an isomorphism class of E_∞-maps from the 'stable spherical Witt group' $\mathbb{S} \otimes W$,

$$\varphi : \mathbb{S} \otimes W \to \mathrm{GL}_1(\mathbb{B}), \tag{1.2}$$

[6] Uniqueness of $\mathbb{V}^\otimes$ as invertible object in $\mathbb{EF}$ follows from the multiplicativity of the Frobenius-Perron dimension.

allowing the construction of a symmetric monoidal crossed-product category

$$(\mathbb{S} \otimes W) \ltimes^{\varphi} \mathbb{F},$$

which we wish to descend to $E'\mathbb{F}$, expressed as an Eilenberg-MacLane crossed product $HW \ltimes \mathbb{F}$.

This descent requires an E_∞ factorization of φ through the Hurewicz map $H : \mathbb{S} \otimes W \to HW$. The obstruction problem thus concerns the mystery arrow m in the diagram

$$\begin{array}{ccccc} & & HW & & \\ & \overset{H}{\nearrow} & & \overset{m}{\searrow} & \\ \mathbb{S} \otimes W & & \xrightarrow{\quad \varphi \quad} & & \mathrm{GL}_1(\mathbb{B}). \end{array} \tag{1.3}$$

The obstruction is a stable k-invariant $k_s^5 \in H_s^5(HW; \mathbb{C}^\times)$, and m admits a torsor of choices over the stable group $H_s^4(HW; \mathbb{C}^\times)$. Using $\mathbb{Z}$, in first instance, instead of W, to identify the obstruction problem, the relevant groups are the stable mapping spaces

$$[H\mathbb{Z}; \Sigma^3 H\mathbb{C}^\times] = 0, \quad [H\mathbb{Z}; \Sigma^4 H\mathbb{C}^\times] = \mathbb{Z}/6, \quad [H\mathbb{Z}; \Sigma^5 H\mathbb{C}^\times] = 0, \tag{1.4}$$

with the middle one generated by the Steenrod operation $Sq^4 \times P_3^1$ on $\mathbb{Z}/2 \times \mathbb{Z}/3$.

We can now use a free resolution

$$F_1 W \to F_0 W \to W$$

to identify the obstruction complex for m with $Sq^4 \times P_3^1$ of the resolution for $\mathbf{R}\mathrm{Hom}\,(W; \boldsymbol{\mu}_6)$. Obstructions to m are thus uniquely removed by passing to the natural extension $e : \widetilde{W} \to W$ by $\boldsymbol{\mu}_6$, for which

$$k_s^5 = \left(Sq^4 \times P_3^1\right) \circ e,$$

solving our obstruction problem of (1.2), and completing the construction of $E\mathbb{F}$, via the diagram

$$\begin{array}{ccccc} & & H\widetilde{W} & & \\ & \overset{H}{\nearrow} & & \overset{\widetilde{m}}{\searrow} & \\ \mathbb{S} \otimes \widetilde{W} & & \xrightarrow{\quad \varphi \quad} & & \mathrm{GL}_1(\mathbb{B}). \end{array} \tag{1.5}$$

Full dualizability follows from the realization of summands as fusion categories over BFCs. The classification of invertibles in Property (v) follows from the uniquness of $\mathbb{V}^\otimes \in \mathbb{F}$ as a Morita-invertible fusion category: an invertible in $\mathbb{F}_w$ trivializes the class $[T(w)]$ in the Witt group. The uniqueness properties (vi) and (vii) are developed in Remarks 1.7 and 1.8 below, relating the center $\boldsymbol{\mu}_6$ of the universal Witt extension with characters of π_3^s. □

1.6 Remark. We do not determine the extension class e: it is what it is. For instance, it is known from [DMNO] that W has no 3-torsion, so the $\boldsymbol{\mu}_3$-component is a split extension.

1.7 Remark (Symmetry). Our homotopy calculation can be made more insightful as follows. The fiber of the Hurewicz morphism $\mathbb{S} \to H\mathbb{Z}$ is the infinite loop space $\Omega^\infty \mathbb{S}_{>0}$. By the Barratt-Priddy-Quillen theorem, this is the classifying space of the homotopy-abelianized symmetric group. The lifting problem (1.5) requires a *trivialization* of φ on $\Omega^\infty \mathbb{S}_{>0}$, that is, of the symmetric group action. Otherwise, we get a *projective* symmetric monoidal structure on $E'\mathbb{F}$; see §8.

The ambiguities for m in (1.5) are also understood in this way. The symmetric group action on powers of U defines a stable map $s_U : \Omega^\infty \mathbb{S}_{>0} \to \Sigma^3 H\mathbb{C}^\times$, the classifying space of the unit fusion category. The Hurewicz fibration $\mathbb{S} \to H\mathbb{Z}$ and the vanishing of the relevant cohomology of $\mathbb{S}$ give

$$[\Omega^\infty \mathbb{S}_{>0}; \Sigma^3 H\mathbb{C}^\times] \cong [\Sigma^{-1} H\mathbb{Z}; \Sigma^3 H\mathbb{C}^\times] = \mathbb{Z}/6,$$

under which the map s_U becomes the generator.

1.8 Remark (Universality). Without the uniqueness requirement in (vii), we could forgo the relation $U^{\otimes 6} \equiv \mathbf{1}$ in $\mathbb{EF}$ and opt for an extension of W by $\mathbb{Z}$ instead of $\mathbb{Z}/6$. The undesirable consequence is to create many instances of each 1/2/3-truncated RT theory, differing only by powers of $U^{\otimes 6}$ on the value of a point. Our construction instead ensures that distinct TQFTs are already distinguishable on their 1/2/3 portion.

1.9 Remark (Decomposition into simples). Define a *simple object* $X \in \mathbb{EF}$ by the condition $\mathscr{T}_X(S^2) = \mathbb{V}$ (triviality of the second center). For $X \in \mathbb{F}$, this agrees with indecomposability of the fusion category; moreover, every object in $\mathbb{F}$ is a direct sum of simples. This generalizes to fusion categories over an invertible object $T \in \mathbb{B}$; in fact, simplicity is detected by forgetting the T-action, because a splitting of the fusion category decomposes its Drinfeld center accordingly. This settles the direct sum decomposition into simples also for the new components of $\mathbb{EF}$.

1.10 Remark ('Fake fusion' calculus). A simple object $X \in \mathbb{EF}$ with center $T := \mathrm{End}(\mathrm{Id}_X)$ lies in the $\mathbb{EF}$-component $\mathbb{F}_T$, where it is represented by $T \otimes U^{\otimes k}$, where T is viewed as fusion category over T, and the U-cofactor modifies the symmetry as explained in Remark 1.7. We have a natural isomorphism of *algebra objects in* $\mathbb{F}$ (with the U-cofactors cancelling out)

$$X \boxtimes X^\vee \equiv T,$$

with their natural actions on $X \in \mathbb{F}_T$. This determines the calculus on X: for instance, the action of a group G on X is equivalent to a homomorphism from G to the 3-group of invertible T-modules. When the latter is a fusion category, that is the (higher) Brauer-Picard group of X [ENO]. We will exploit this when discussing orientability (Theorem 5).

More generally, if X_1 and X_2 are objects with centers T_1 and T_2, then $\mathrm{Hom}_{\mathbb{EF}}(X_1, X_2) = 0$, unless T_1 and T_2 are Witt-equivalent and the X_i sit over the same point in the $\boldsymbol{\mu}_6$-extension torsor over $[T_1] = [T_2]$. In the latter case, an equivalence $T_1 \boxtimes T_2^{rev} \equiv Z(F)$ identifies the Hom 2-category with that of semi-simple F-module categories.

1.11 Remark (k-invariant of $\mathrm{GL}_1(\mathbb{EF})$). The (higher) group of units of $\mathbb{EF}$ has the two non-zero homotopy groups $\pi_0 = \mathbb{Z}/6$, $\pi_3 = \mathbb{C}^\times$, related by the universal k-invariant $Sq^4 \times P_3^1$. More precisely, in terms of the co-fiber $\mathbb{S}/\eta$ in the sequence

$$\mathbb{S}^1/2 \overset{\eta}{\rightarrowtail} \mathbb{S} \twoheadrightarrow \mathbb{S}/\eta,$$

the spectrum $\mathrm{GL}_1(\mathbb{EF})$ is the connective cover of the 3-*shifted Pontrjagin dual* $\mathrm{Map}\left(\mathbb{S}/\eta; \Sigma^3 \mathbb{I}_{\mathbb{C}^\times}\right)$.

The class η represents the Koszul sign rule in the symmetric tensor structure on super-vector spaces, and killing it reflects the fact that we do not allow odd vector spaces in our target. This gives credence to $\mathbb{EF}$ as the universal target for *bosonic* 3-dimensional TQFTs.

1.12 Remark (4 dimensions). Adding fake fusion categories from $\mathbb{EF}$ as 1-morphisms to the 4-category $\mathbb{B}$ kills the Witt group, and produces a 4-category with spectrum of units $B\mathrm{GL}_1(\mathbb{EF})$. This matches the (4-shifted) Pontrjagin dual of $\mathbb{S}/\eta$: $\pi_4\left(\mathbb{S}/\eta\right) = 0$. However, this is not quite a universal target for 4-dimensional bosonic TQFTs, as one can reasonably add objects to $\mathbb{B}$, such as the fusion 2-categories of [CF2], and their generalizations using objects from $\mathbb{EF}$, as illustrated in Apppendix A. Our construction applies equally well to such enlargements of $\mathbb{B}$, with their groups of invertible isomorphism classes replacing W. We plan to return to this in a follow-up paper.

2. Main result: fermionic case

The main theorem has a natural generalization when super-vector spaces are included. Recall that the tensor structure on the category $S\mathbb{V}^{\otimes}$ of super-vector spaces is symmetric, under the Koszul sign rule. This defines an (also symmetric) tensor structure on the 2-category $S\mathbb{L}$ of finite semi-simple module categories over $S\mathbb{V}^{\otimes}$.

2.1 Definition. A finite, semi-simple *super-category* is one equivalent to the category of modules (in super-vector spaces) over a finite complex semi-simple super-algebra. *Functors* between super-categories are required to be linear over $S\mathbb{V}^{\otimes}$. Tensor products are taken over $S\mathbb{V}^{\otimes}$. A *fusion super-category* is an $S\mathbb{V}^{\otimes}$-algebra in $S\mathbb{L}$ in which all objects have internal left and right duals. Braidings and $S\mathbb{B}$ are defined as expected.

2.2 Remark. We refer to [FT...] for a wider discussion, but summarize the main points here:

- (i) Finite-dimensional, complex, simple algebras in super-vector spaces are Morita equivalent to one of $\mathbb{C}$ or $\mathrm{Cliff}(1)$. A simple object in a super-category in $S\mathbb{L}$ therefore generates an additive summand of either $S\mathbb{V}$ or of the regular $\mathrm{Cliff}(1)$-module.
- (ii) Just as in the bosonic case, we use 'fusion' where some authors use 'multi-fusion'; every indecomposable fusion super-category is isomorphic (in $S\mathbb{F}$) to one with simple unit.
- (iii) Indecomposable fusion super-categories have non-degenerate Drinfeld centers, and every fusion super-category splits into a direct sum of indecomposables.
- (iv) A fusion super-category F cannot be pivotal if it include $\mathrm{Cliff}(1)$-lines. Moreover, its Drinfeld center $\mathrm{End}_{F\text{-}F}(F)$ and co-center $F\boxtimes_{F\boxtimes F^{op}} F$ can be inequivalent categories.
- (v) As a result, Spin structures are needed in the respective TQFTs.

Theorem 2. *There exists a symmetric monoidal* 3*-category* $\mathrm{E}S\mathbb{F} \equiv \bigoplus_{w\in\widetilde{SW}} S\mathbb{F}_w$ *satisfying (the super analogues of) properties (i)-(vii) in §0.6, and it is unique up to isomorphism.*

Proof. The argument is the same, *mutatis mutandis*. The principal change is that the group $\mathrm{GL}_1(S\mathbb{F})$ acquires two new homotopy groups $\pi_2 = \pi_1 = \mathbb{Z}/2$, in addition to $\pi_3 = \mathbb{C}^\times$ at the top, from the odd lines and odd Clifford algebras, respectively. Specifically, $\mathrm{GL}_1(S\mathbb{F})$ it is the connected cover of the shifted Pontrjagin dual to $\mathbb{S}$:

$$\mathrm{GL}_1(S\mathbb{F}) = \left(\Sigma^3\mathbb{I}_{\mathbb{C}^\times}\right)_{>0}.$$

There is a similar higher group $\mathrm{GL}_1(S\mathbb{B})$, having as π_0 the super-Witt group SW of invertible isomorphism classes in $S\mathbb{B}$. The obstruction problem is defined by the same diagram (1.5), with $SW, S\mathbb{F}, Sm$. However, the relevant homotopy groups are now

$$\left[H\mathbb{Z};\left(\Sigma^3\mathbb{I}_{\mathbb{C}}^{\times}\right)_{>0}\right] = 0,\quad \left[H\mathbb{Z};\left(\Sigma^4\mathbb{I}_{\mathbb{C}}^{\times}\right)_{>1}\right] = \mathbb{Z}/24,\quad \left[H\mathbb{Z};\left(\Sigma^5\mathbb{I}_{\mathbb{C}}^{\times}\right)_{>2}\right] = 0. \tag{2.3}$$

Indeed, we can determine them from the fibration sequence

$$\Sigma^{k-4}H\mathbb{Z}/24 \rightarrowtail \left(\Sigma^k\mathbb{I}_{\mathbb{C}}^{\times}\right)_{>k-3} \twoheadrightarrow \left(\Sigma^k\mathbb{I}_{\mathbb{C}}^{\times}\right)_{>k-6},$$

which holds for all $k\in\mathbb{Z}$, because of the vanishing $\pi_4^s = \pi_5^s = 0$. Combining this for $k = 3,4,5$ with the vanishing of $[H\mathbb{Z};\Sigma^k\mathbb{I}_{\mathbb{C}^\times}]$ for $k>0$, we are led from (2.3) to the groups

$$\left[H\mathbb{Z};\Sigma^{-1}H\mathbb{Z}/24\right] = 0,\quad [H\mathbb{Z};H\mathbb{Z}/24] = \mathbb{Z}/24,\quad [H\mathbb{Z};\Sigma H\mathbb{Z}/24] = 0.$$

As before, a free resolution $F_1SW \to F_0SW \to SW$ converts the obstruction complex for our desired lifting of $S\varphi$ in (1.5) into the resolution of $\mathbf{R}\mathrm{Hom}\,(W;\mathbb{Z}/24)$. The obstruction problem is canonically resolved by a central extension

$$\mathbb{Z}/24 \rightarrowtail \widetilde{SW} \twoheadrightarrow SW,$$

allowing us to define

$$\mathrm{E}S\mathbb{F} = \bigoplus\nolimits_{w\in\widetilde{SW}} (S\mathbb{F})_{T(w)}.$$

The universal properties are seen in the same way as for $\mathrm{E}\mathbb{F}$. □

(2.4) Invertible TQFTs. The new $\mathrm{E}S\mathbb{F}$ contains 24 isomorphism classes of invertibles, versus only the unit $S\mathbb{V}^{\otimes}$ in $S\mathbb{F}$. The invertibles represent the 24 invertible framed TQFTs in dimension 3, matching the 24 distinct possible symmetric monoidal structures on an invertible object. The center $\boldsymbol{\mu}_{24}$ of $\widetilde{SW}$ is naturally identified with the Pontrjagin dual of π_3^s, and the generating theory ψ (see §3.9), which defines the standard inclusion $\pi_3^s \subset \mathbb{C}^\times$, is characterized by a symmetry akin to that for U, in Remark 1.7.

3. Anomalous TQFTs and reduced central charge

We discuss the *anomalous* versions for the TQFTs $\mathscr{T}_X$ defined by objects $X \in \mathrm{E}S\mathbb{F}$, their linearizations in tangential structures, and their central charges. Appendix B contains a wider discussion of tangential structures. We focus on Spin^{rp_1}-structures, with tangent bundles classified by the homotopy fibers of $rp_1 : B\mathrm{Spin}(3) \to \Sigma^4 H\mathbb{Z}$ ($r = 1, 1/2$ or $1/4$) and their dimensionally fixed variants. Oriented p_1-structures are discussed in the next section.

(3.1) Action of $\mathrm{Spin}(3)$ *via* μ. The Cobordism Hypothesis yields a change-of-framing action of $\mathrm{O}(3)$ on the 3-dualizable objects and morphisms in the 3-category $\mathrm{E}S\mathbb{F}$. In our case, those assemble to the full underlying groupoid $\iota_{>0}\mathrm{E}S\mathbb{F}$, because every object is 3-dualizable [BJS]. The absence of lower homotopy groups forces the subgroup $\mathrm{Spin}(3)$ to act via $\pi_3 = \mathbb{Z}$, defining a (necessarily scalar) 3-automorphism $\mu(X)$ on each simple object X. A unit local change in 3-framing acts via the generator of $\pi_3\mathrm{Spin}(3)$, and we conclude

3.2 Theorem. *In the* 3*-framed TQFT* $\mathscr{T}_X$ *defined by a simple object* X*,* 3*-morphisms transform by* $\mu(X)$ *under a unit local change in* 3*-framing. Lower morphisms are unchanged up to isomorphism.* □

The scalar $\mu(X)$ is multiplicative in X, and is the projective obstruction to $\mathrm{Spin}(3)$-invariance: a simple object $X \in \mathrm{E}S\mathbb{F}$ is invariant precisely when $\mu(X) = 1$. In the next section, we show this to be so when $X = F \in S\mathbb{F}$; for fusion categories, this was conjectured in [DSPS]. Then, $\mathscr{T}_F$, *a priori* defined on 3-framed manifolds, factors uniquely through Spin-manifolds.

3.3 Proposition. *μ surjects the center $\boldsymbol{\mu}_{24} \subset \widetilde{SW}$ onto the 12^{th} roots of unity $\boldsymbol{\mu}_{12}$.*

Proof. In the Atiyah-Hirzebruch spectral sequence computing $\mathbb{I}^*_{\mathbb{C}^\times} MT\mathrm{Spin}(3)$, the differential

$$d_4 : (\pi_3^s)^\vee \to H^4\big(B\mathrm{Spin}(3);\mathbb{C}^\times\big) \cong \mathbb{C}^\times,$$

which describes μ on the center of $\widetilde{SW}$, has kernel $\mathbb{Z}/2$ and image $\boldsymbol{\mu}_{12}$. □

(3.4) Anomalous and linearized theories. For general X, the projective Spin invariance allows a presentation of $\mathscr{T}_X$ as an *anomalous* TQFT for Spin manifolds (a structure we recall momentarily). This anomalous presentation may be traded back for a p_1-related tangential structure plus a specified transformation law. Let $\alpha_{\mu(X)}$ (or simply α_X) denote the 4-dimensional invertible TQFT, defined on manifolds with Spin(3) structure, valued in the spectrum $\Sigma^4\mathbb{I}_{\mathbb{C}^\times}$, and characterized by the closed manifold invariant $M \mapsto \mu(X)^{p_1(M)/4}$. The following summarizes the anomaly/tangential trade.

Theorem 3. *The object $X \in \mathrm{ES}\mathbb{F}$ defines an* anomalous 3-dimensional Spin theory $\alpha\mathscr{T}_X$, *a boundary theory for the anomaly theory α_X. We can linearize $\alpha\mathscr{T}_X$ as follows:*

- *(i) $\alpha\mathscr{T}_X$ is linearizable in two ways over* $\mathrm{Spin}^{p_1/4}(3)$*-manifolds (framed) such that one step in $p_1/4$-structure changes 3-morphisms by a factor of $\mu(X)$.*
- *(ii) After a choice of $\mu(X)^{1/2}$, $\alpha\mathscr{T}_X$ is linearizable in two ways over* $\mathrm{Spin}^{p_1/2}(3)$*-manifolds (stably framed) such that one step in $p_1/2$ structure changes 3-morphisms by a factor of $\mu(X)^{1/2}$.*
- *(iii) After a choice of $\mu(X)^{1/4}$, $\alpha\mathscr{T}_X$ is linearizable in two ways over* $\mathrm{Spin}^{p_1}(3)$*-manifolds. One step in p_1 structure changes 3-morphisms by a factor of $\mu(X)^{1/4}$.*

In (iii), the anomaly theory α_X factors uniquely through oriented 4-manifolds.

3.5 Remark. The trade carries cost (if X is forgottten), because of the automorphisms of α_X. These can be described in terms of the invertible theories ν, ω, and ψ described in §3.9 below. For instance, attempting to reconstruct $\mathscr{T}_X$ from $\alpha\mathscr{T}_X$ loses the distinction between X and $XU^{\otimes 3}$, even though the two define distinct framed theories. More generally,

- (i) The two choices for $\mathscr{T}_X$ in (i-iii) differ by a factor of ν.
- (ii) In (ii), tensoring with ω flips the choice of square root of $\mu(X)$.
- (iii) In (iii), we can cycle through choices of fourth root by powers of $\psi^{\otimes 12}$.
- (iv) The structures (ii-iii) involve lifting $\underline{c}$ modulo 12 and 24, respectively.

3.6 Remark. The *reduced central charge* $\underline{c} := \frac{6}{2\pi i}\log\mu \pmod{6\mathbb{Z}}$ makes the theory α_X more familiar:

$$\mu(X)^{p_1/4} = \exp\left(2\pi i\underline{c}(X)\cdot\frac{p_1}{24}\right).$$

Recall that $p_1/4$ identifies $\pi_4 B\mathrm{Spin}(3)$ with $\mathbb{Z}$.

(3.7) Refresher on anomalous theories. Anomalous TQFTs may be described as boundary theories for an invertible theory in one higher dimension; see for instance [FT3]. Invertible TQFTs map into into the spectrum of units of the target category, and thus factor through *stable* maps from the monoidal group completions of the (tangentially appropriate) bordism categories, the Madsen-Tillmann spectra.[7] When the units in the target category form (the connective cover of) the spectrum $\Sigma^4\mathbb{I}_{\mathbb{C}^\times}$, stable maps with that target are classified by the Pontrjagin dual of π_4 of the source. The invertible TQFT is then determined by the numerical invariants of the TQFT on closed manifolds. We refer to [FH] for the comprehensive account of these ideas. Our anomaly theory α_X is defined, in first instance, on 4-manifolds with Spin(3) structure, where $p_1/4$ is an integral class.

The natural home (target 4-category) to use for α_X is the delooping $B\mathrm{ES}\mathbb{F}$: a symmetric monoidal category with a single object[8] $\mathbf{1} = \mathrm{ES}\mathbb{F}$. The group of units is precisely the connective cover of $\Sigma^4\mathbb{I}_{\mathbb{C}^\times}$. The point generator for α_X is the base object, making it trivial as a *framed* theory, and the anomalous theory is X itself, as a morphism from $\mathbf{1}$ to itself, but with the source and target carrying different Spin(3)-invariance data: the trivial and interesting one.

[7] Due to standard conventions, the group-completions are the deloopings of MT spectra which start in degree 0.
[8] To give this a semblance of respectability, we could also include direct sums of copies of $\mathbf{1}$.

3.8 Remark. This delooping target works for any anomaly theory, but is somewhat unsatisfactory: we wish to land in (an enhancement of) $S\mathbb{B}$, a universal target for 4-dimensional TQFTs defined from suitable algebra objects in $\mathbf{E}S\mathbb{F}$. There, the point generator of α_X is represented by the algebra object $\mathrm{End}(X)$ in $\mathbf{E}S\mathbb{F}$; its anomaly for Spin(3)-invariance is naturally cancelled. The object X then defines a boundary theory $\alpha\mathscr{T}_X$ for α_X—the standard module for its own endomorphism algebra. We plan to develop this approach in [FST].

(3.9) The invertibles ν, ω and ψ. Note first that, on a 3-manifold, a $\mathrm{Spin}^{p_1/4}(3)$-structure is equivalent to a 3-*framing*, while a $\mathrm{Spin}^{p_1/2}(3)$ structure is equivalent to a *stable framing*. This is because $B\mathrm{Spin}^{p_1/4}(3)$ is 4-connected, whereas $B\mathrm{Spin}^{p_1/2}(3)$ agrees with the homotopy fiber of $B\mathrm{SO}(3) \to B\mathrm{SO}$ through dimension 4. Going further, $\mathrm{Spin}^{p_1}(3)$-structures are a step towards the $\mathrm{Spin}^{\mathbb{C}p_1}(3)$-structures of §7. The reader may wish to consult Appendix B and the bordism groups in Table 2.

Promoting 3-framed theories to p_1-tangential structures meets some invertible ambiguities:

(i) The 3-*framed* theory ν is determined by its value $\nu(pt) = U^{\otimes 3} \in \mathbf{E}\mathbb{F}$, the unit of order 2. However, it factors uniquely through Spin(3) manifolds, because the sign character of π_3^s is the natural map

$$\nu : \pi_3^s \to \pi_0 MT\mathrm{Spin}(3) \cong \mathbb{Z}/2.$$

Alternatively, $\mu(U^{\otimes 3}) = 1$, so the Spin(3) action is trivial; factorization follows from the Cobordism Hypothesis. (We describe the associated Spin manifold invariant in §B.5.)

(ii) *Stably framed* 3-manifolds carry an invertible order-two theory ω, which detects the difference between the two trivializations of w_4: from the 3-dimensional structure, and from the stable framing.

Unlike ν, the theory ω is trivial on 3-*framed* manifolds, where the two cancellations of w_4 agree. In particular, $\omega(pt) = \mathbf{1} \in \mathbb{F}$. On the other hand, ω detects a unit shift in $p_1/2$ structure, whereas the lift of ν to $\mathrm{Spin}^{p_1/2}(3)$-manifolds is insensitive to that shift.

(iii) On framed manifolds, the standard character $\pi_3^s \to \mathbb{C}^\times$ defines the *free fermion theory* ψ. Thus, $\psi(pt) \in \mathbf{E}S\mathbb{F}$ is the invertible object sitting over the generator of $\ker(\widetilde{SW} \to SW)$, and $U = \psi(pt)^{\otimes 4}$. We extend ψ invertibly to $\mathrm{Spin}^{p_1}(3)$-manifolds using the numerical invariant

$$\psi : \pi_0 MT\mathrm{Spin}^{p_1}(3) \cong \pi_3 M\mathrm{Spin}^{p_1} \oplus \pi_0 MT\mathrm{Spin}(3) \to \pi_3 M\mathrm{Spin}^{p_1} \cong \mathbb{Z}/48 \rightarrowtail \mathbb{C}^\times,$$

coming from the projection followed by the standard character. We will further extend ψ to $\psi_{\mathbb{C}}$ on $\mathbb{C}p_1$-structures in §7, using the analogous splitting of $\pi_0 MT\mathrm{Spin}^{\mathbb{C}p_1}(3)$.

The theories ν, ω, ψ and $\mathscr{T}_U$ are related as follows:

- $\omega = \nu \otimes \psi^{\otimes 12}$ on stably framed manifolds.
  This formula extends ω to $\mathrm{Spin}^{p_1}(3)$-manifolds, and to $\mathbb{C}p_1$-structures via $\psi_{\mathbb{C}}$, but then it no longer has order 2.
- $\mathscr{T}_U = \nu \otimes \psi^{-8}$ on $\mathrm{Spin}^{p_1}(3)$-manifolds, where we define $\mathscr{T}_U$ on $\mathrm{SO}^{p_1}(3)$-manifolds by standard character $\pi_0 MT\mathrm{SO}^{p_1}(3) = \mathbb{Z}/6 \to \mathbb{C}^\times$, and then lift to Spin.
  As before, this formula extends $\mathscr{T}_U$ to $\mathbb{C}p_1$-structures, but it no longer has order 6.

Upon extension to $\mathrm{Spin}^{p_1}(3)$-structures, the mod 6 reduced central charges $\underline{c}$ refine mod 24, reflecting the behavior under a one-unit shift of p_1-structure. Further extension to $\mathbb{C}p_1$-structures (cf. §7) will lift the central charges to $\mathbb{C}$. The relations above then tell us that

$$c(\nu) = 0, \quad c(\psi) = \frac{1}{2}, \quad c(\omega) = 6, \quad c(\mathscr{T}_U) = -4 \pmod{24}.$$

(3.10) Caution. We stress that the central charge c is defined in terms of coupling to (multiples of) p_1, not to (re)framings. If this is not tracked correctly, the kernel $\{\pm 1\}$ of μ (Proposition 3.3) creates conflicts. For instance, the relation $U = \psi(pt)^{\otimes 4} \in \mathrm{E}S\mathbb{F}$ could suggest that $c(\mathscr{T}_U) = 2$. Of course, $2 = -4 \pmod 6$, matching the answers on 3-framed theories, but trouble comes from assuming that a unit shifts in π_3^s and in $p_1/2$ have the same effect on stable framings. The theory ν breaks that link, and the maps in Proposition B.11 show that $\mathscr{T}_U \neq \psi^{\otimes 4}$ as $\mathrm{Spin}^{p_1}(3)$-theories. Section B.12 shows other apparent inconsistencies around ψ, if its domain is not tracked carefully.

(3.11) Loss of information. As already mentioned, the anomalous version $\alpha\mathscr{T}_X$ is less precise than $\mathscr{T}_X$, since α_X can be trivialized more often than first meets the eye. Thus,

3.12 Proposition. *When $\underline{c}(X) \in \frac{1}{2}\mathbb{Z}$, $\alpha\mathscr{T}_X$ can be linearized to a* Spin *TQFT in two ways, differing by ν.*

Proof. When $\mu(X) \in \boldsymbol{\mu}_{12}$, Rohlin's theorem ensures that $\mu(X)^{p_1(M)/4}$ vanishes on closed Spin 4-manifolds, so that $\alpha_{\mu(X)}$ is trivializable; the trivializations form a torsor over $\langle \nu \rangle$. □

When $\underline{c}$ is *half*-integral, there is no preferred choice; but a preferred one for integral $\underline{c}$ is obtained by squaring either choice of trivialization for $\underline{c}/2$.

4. Spin invariance of fusion super-categories.

We now show that the invariant μ factors through the (centally extended) super-Witt group. The reader may also want to consult the proof of the closely related orientation Theorems 5 and 6 for a parallel argument in a more familiar (modular) context.

Theorem 4. *The action of* $\mathrm{Spin}(3)$ *on* $\mathrm{E}S\mathbb{F}$ *factors through a group homomorphism $\mu : \widetilde{SW} \to \mathbb{C}^\times$. In particular, a fusion super-category $F \in S\mathbb{F}$ has $\mu \equiv 1$, and carries unique* $\mathrm{Spin}(3)$*-invariance data.*

4.1 Remark. The condition $[X] = [Y] \in \widetilde{SW}$ is equivalent to $X \boxtimes Y^\vee \in S\mathbb{F}$, so the special case is equivalent to the general statement.

Proof. This would be straightforward, if the Spin(3)-action extended to all 1-morphisms in $\mathrm{E}S\mathbb{F}$: indeed, π_3 would act trivially on 1-morphisms, due to their categorical cutoff, and a fusion super-category F is related to the unit $S\mathbb{V}^\otimes$ by the regular module ${}_FF$, forcing the equality of projective obstructions: $\mu(F) = \mu(S\mathbb{V}^\otimes) = 1$.

While Spin(3) need not *a priori* act on on the collection of *all* 1-morphisms in $S\mathbb{F}$, we will use the equivalence $\mathrm{Spin}(2)/\Omega S^2 \sim \mathrm{Spin}(3)$ from the Hopf fibration to read off μ from Spin(2), which *does* act on $\iota_{>1}\mathrm{E}S\mathbb{F}$. Every fusion super-category F is naturally invariant under ΩS^2, because the action of that group on $\iota_{>0}\mathrm{E}S\mathbb{F}$ factors through the trivial map to Spin(3). Thus, ΩS^2 acts on $\mathrm{Hom}_{S\mathbb{F}}(S\mathbb{V}^\otimes; F)$. We claim that the regular module ${}_FF$ therein carries a natural ΩS^2-invariance structure. This extends the action of Spin(2) to Spin(3), and completes the argument.

For our claim, it suffices to trivialize the action of $\pi_1\Omega S^2$ on ${}_FF$, compatibly with its natural trivialization on F: there are no further obstructions to a section through ${}_FF$ over $S^2 = B\Omega S^2$. Now, $\pi_1\Omega S^2$ acts on F and ${}_FF$ by their squared *Serre automorphisms*. Invariance is the content of the following addition to Theorem 4; this extends the action of Spin(3) to 1-morphisms in $\mathrm{E}S\mathbb{F}$ (albeit not in a way compatible with composition). □

4.2 Theorem. *For $F \in S\mathbb{F}$, every F-module $M \in \mathbb{L}$ admits a trivialization of the square of its Serre automorphism S_M relative to F, compatible with the* $\mathrm{Spin}(3)$*-enforced trivialization of S_F^2 on F.*

4.3 Remark. Underlying the theorem is a (potentially stronger) property internal to F, which indeed is what we prove. The square of the Serre functor on $F \in S\mathbb{F}$ is the quadruple dual, identifiable with Id_F *as a tensor functor* using Radford's isomorphism [DSPS]. Any other identification $S_F^2 \cong \mathrm{Id}_F$ as automorphisms of $F \in S\mathbb{F}$ differs from Radford's by braiding with a central element $z \in Z(F)$. The content of the theorem is that z maps to $\mathbf{1} \in F$ for the identification enforced by the projective Spin(3)-invariance of F (via $\pi_1\mathrm{SO}(3) = \mathbb{Z}/2$). The same identification, acting on the regular module ${}_FF$, then gives our compatible trivialization of S_F^2 there. The case of other modules reduces to the regular one by Morita equivalences.

Proof. View the Serre functor S_F of F as a co-oriented self-interface in a (locally) constant framing for F, performing a full framing twist upon crossing the interface. This is not a genuine framing defect, as it can be spread out into the bulk by a deformation; but treating it as such facilitates the argument. The framing jump across this interface can be ended in a genuine framing singularity, described by a tangent vector along the supporting line plus the radial framing in the normal directions. (Our TQFT functor $\mathscr{T}_F$ will remain *undefined* on this framing defect. We will only need to evaluate it when investigating SO(3)-invariance later.)

The squared defect S_F^2 is also endable in a framing defect, now with a double-twist (dipole) singularity in the normal directions. This double twist is implemented by a big circle in Spin(3), trivializable by a contracting homotopy which we choose once and for all.[9] This defines an ending defect ∂S_F^2 for S_F^2 in $\mathscr{T}_F$, which on any linking circle is isomorphic to the unit object $\mathbf{1} \in Z(F)$, the *transparent defect*.

On the regular boundary theory (which we keep denoting M for notational clarity), the bulk Serre interfaces S_F, S_F^2 can be terminated in interfaces S_M and S_M^2, implementing a (now tangential) surface Serre framing twist and its square. Trivializing S_M^2 compatibly with ∂S_F^2 means ending it in an invertible defect, which will be a boundary endpoint of ∂S_F^2, as in Figure 1:

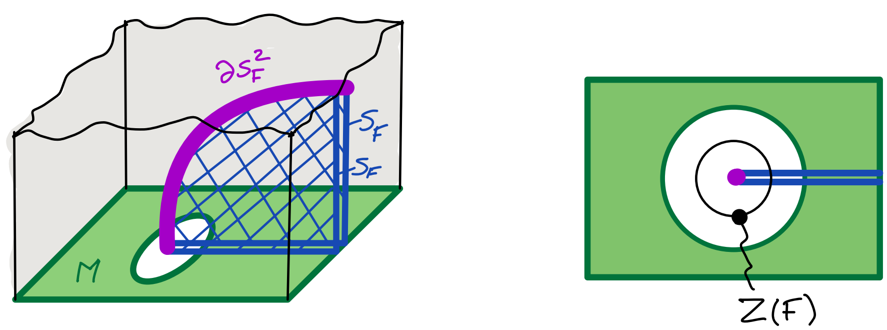

Figure 1: The end ∂S_F^2 of the bulk squared-Serre reaching the boundary and its value in $Z(F)$

We cannot end the defect *geometrically*: the framing must *a priori* be tangent to the boundary, which precludes a trivialization of the double twist. However, the end ∂S_F^2 of S_F^2 produces the vector $\mathbf{1} \in Z(F)$ in a linking disk, so our desired ending defect lives in the $\mathscr{T}_F$-space produced by the clean disk with M-boundary and doubly twisted boundary 2-framing. The standard elbow of a solid cylinder with M-boundary identifies this with the dual of the same picture, but with the blackboard framing. If the fusion category F is simple with simple unit — which we may arrange,

[9]Note that the normal bundle to the line is framed by the Serre interface. Without the latter, the defect trivialization is ambiguous along a loop, because the space of contracting homotopies is a torsor over $\Omega^2 S^3$. The p_1-structure pins the ambiguity.

for purposes of this theorem, by Morita equivalences and direct sum decompositions — then this picture computes the vector space

$$\mathbb{C} \cong \mathrm{Hom}_{Z(F)}(\mathbf{1}, W) = \mathrm{Hom}_F(\mathbf{1}, \mathbf{1})$$

with the 'Wilson object' $W = \iota^*(\mathbf{1})$ in $Z(F)$ ($\iota : Z(F) \to F$ is the natural morphism). A non-zero vector therein end must define an isomorphism between Id_M and S_M^2, as both are invertible, and therefore simple objects in $\mathrm{End}_F(M)$. □

5. Orientability and modular structures

When $X \in \mathbb{EF}$ is supplemented by a modular tensor structure on $\mathscr{T}_X(S_b^1)$ (for the circle with bounding 3-framing), we can forgo the Spin structure and factor $\mathscr{T}_X$ through *oriented* p_1-manifolds. Now, the anomaly theory α_X in Theorem 3.iv has 6 trivializations as an $\mathrm{SO}^{p_1}(3)$-theory, related by the powers of U. This ambiguity in reconstructing $\mathscr{T}_X$ from Walker's anomalous model is removed by remembering the point generator X.

Theorem 5. *For $X \in \mathbb{EF}$, a modular structure on the braided fusion category $T := \mathscr{T}_X(S_b^1)$ descends $\mathscr{T}_X$ uniquely to oriented manifolds with p_1-structure. Thereby, $\mu(X)$ acquires a preferred fourth root.*

5.1 Remark. For 1/2/3-manifolds, Turaev's construction of TQFTs *with signature structure* is a version of this result; an account is also found in [BDSV]. Interpretation requires some care, though: Turaev's construction can only be refined to a symmetric monoidal functor between *projectively symmetric* monoidal categories (see our discussion in §8). A sign ambiguity in that construction comes from tensoring with U, which has central charge (-4). Knowledge of X and the use of p_1-structures refines the central charge mod 24.

(5.2) Preliminaries. When a connected group G acts on a topological space S, a choice of base-point $x \in S$ leads to a group extension

$$\Omega_x(S) \rightarrowtail \Omega_x(S_G) \twoheadrightarrow G, \tag{5.3}$$

as part of the fibration sequence that continues with

$$G \xrightarrow{\cdot x} S \to S_G \to BG. \tag{5.4}$$

Invariance data for x is a homotopy G-fixed point structure: a section of the map $S_G \to BG$ through the point x. This is equivalent to a section of the fibration (5.3) *as a group homomorphism*.

We will produce invariance data for $X \in \mathbb{EF}$ under the group $\mathrm{SO}^{p_1}(3)$ for precisely one choice of $\mu(X)^{1/4}$. Recall that $\mathrm{SO}^{p_1}(3)$ is the fiber of $\Omega p_1 : \mathrm{SO}(3) \to K(\mathbb{Z}; 3)$. Given the absence in $\mathbb{EF}$ of $(k > 3)$-morphisms, we may Postnikov-truncate to the group G with only

$$\pi_1 G = \mathbb{Z}/2, \qquad \pi_2 G = \mathbb{Z}/4,$$

extended in BG by the *Pontryagin square* $\wp : K(\mathbb{Z}/2; 2) \to K(\mathbb{Z}/4; 4)$.

Proof of Theorem 5. On $S = \iota_{>0}\mathbb{EF}$, the group G acts via $\mathrm{O}(3)$; with $x = X$ in the above discussion and writing GA_X for $\Omega_x(S_G)$, we get an extension

$$\mathrm{Aut}_{\mathbb{EF}}(X) \rightarrowtail GA_X \twoheadrightarrow G, \tag{5.5}$$

and a G-fixed point structure on X is a group splitting $G \to GA_X$ of the last map.

The group $\mathrm{Aut}_{\mathbb{EF}}(X)$ is determined by $T = \mathscr{T}_X(S^1_b)$ (Remark 1.10) and has homotopy groups

$$\pi_0 \,\mathrm{Aut}_{\mathbb{EF}}(X) = [\mathrm{Aut}^{br} T], \quad \pi_1 = [\mathrm{GL}_1 T], \quad \pi_2 = \mathbb{C}^\times, \tag{5.6}$$

(*isomorphism classes* of braided tensor automorphisms, invertibles, and scalars). The group GA_X is the homotopy quotient

$$\Omega G \rightarrowtail \mathrm{Aut}_{\mathbb{EF}}(X) \twoheadrightarrow GA_X.$$

By construction of G, the action of $K(\mathbb{Z}/4;2) \subset G$ on $\iota_{>0}\mathbb{EF}$ factors as the (stable) map

$$K(\mathbb{Z}/4;2) \xrightarrow{B_4} K(\mathbb{Z};3) \xrightarrow{\mu} K(\mathbb{C}^\times;3), \tag{5.7}$$

ensuring the vanishing of the map

$$\mathbb{Z}/4 = \pi_1\Omega G \to \pi_1\mathrm{Aut}_{\mathbb{EF}}(X).$$

If the map $\pi_0\Omega G \to [\mathrm{Aut}^{br} T]$ is also trivial, as needed for a splitting of (5.5), then the homotopy groups of GA_X are

$$\pi_0 GA_X = [\mathrm{Aut}^{br} T],$$

while π_1 and π_2 are the group extensions induced by the map $GA_X \to G$ in (5.5):

$$[\mathrm{GL}_1 T] \rightarrowtail \pi_1 GA_X \twoheadrightarrow \pi_1 G = \mathbb{Z}/2, \tag{5.8}$$

$$\mathbb{C}^\times \rightarrowtail \pi_2 GA_X \twoheadrightarrow \pi_2 G = \mathbb{Z}/4. \tag{5.9}$$

The second one, classified by (5.7), has four splittings, matching the four choices of $\mu(X)^{1/4}$.

A group splitting of (5.5) is then equivalent to

(i) the vanishing of the connecting map $\pi_1 G \to [\mathrm{Aut}^{br} T]$ in (5.5),
(ii) a splitting of the consequent group extension (5.8) of $\pi_1 G$,
(iii) a splitting of the resulting extension of G by $B^2\mathbb{C}^\times$.

Now, once (i) and (ii) have been addressed, (iii) has a unique solution, because

$$H^3(BG;\mathbb{C}^\times) = H^4(BG;\mathbb{C}^\times) = 0. \tag{5.10}$$

More specifically, a map $K(\mathbb{Z}/2;2) \to K(\mathbb{C}^\times;4)$, classifying a group extension of the base $B\mathbb{Z}/2$ of G by $B^2\mathbb{C}^\times$, factors uniquely through the Pontryagin square $\wp$ to $K(\mathbb{Z}/4;4)$, the k-invariant of BG. Every extension of G by $B^2\mathbb{C}^\times$ is then split by exactly one of the four splittings of (5.9). The choice of fourth root of $\mu(X)$ therein also decouples the actions of the two Pontrjagin layers of G, after we push out $\pi_2 G$ into $B^3\mathbb{C}^\times$ through $\mu(X) \circ B_4$. We must then only handle items (i) and (ii) above, and they only concern $\pi_1 G = \pi_1\mathrm{SO}(3) = \mathbb{Z}/2$.

We refer to [BK, DSPS, HPT, P] for basic results on braided fusion categories. On X, $\pi_1\mathrm{SO}(3)$ acts by the Serre automorphism S_X, equipped with a trivialization of its square. In $\mathrm{Aut}^{br} T$, this becomes the square of the braiding β: the identity functor on T with a braided automorphism of the multiplication.[10] This functor, and the connecting map in (i) along with it, is trivialized by a *balancing twist*: an automorphism θ of the identity of the underlying category of T, related to the braiding β by the identity

$$\theta_{a\otimes b} \circ \left(\theta_a^{-1} \otimes \theta_b^{-1}\right) = \beta_{b,a}\beta_{a,b}, \quad \forall a, b \in T.$$

[10] Drinfeld's isomorphism of objects with their double duals identifies S with the internal double dual functor (the Serre functor on $\mathscr{T}_X(S^1)$ qua fusion category).

From θ, we extract a *tensor* automorphism of Id_T:

$$\rho(\theta) : a \mapsto \theta_a \circ (\theta^*_{a^*})^{-1} \in \mathrm{End}(a), \quad \forall a \in T.$$

Non-degeneracy of β ensures that ρ is effected by the double-braiding with some $t \in \mathrm{GL}_1 T$:

$$W(t) := (t^{-1}\otimes) \circ (\beta_{x,t}\beta_{t,x}) \circ (t\otimes) : x \xrightarrow{\sim} x. \tag{5.11}$$

The isomorphism class of $t \in \mathrm{GL}_1 T$ modulo squares represents the extension class in (5.8). If t has a square root r, we can kill ρ by composing θ with $W(r)^{-1}$.

The *ribbon condition* on θ, the final modularity constraint for $T = \mathscr{T}_X(S^1_b)$, is precisely $\rho \equiv 1$, so such a θ provides the splitting required in (ii). □

5.12 Remark. The complement $\pi_1 GA_X \setminus [\mathrm{GL}_1 T]$ can be identified with the set of balancings: a torsor over $[\mathrm{GL}_1 T]$ under composition with W. Its addition law into $[\mathrm{GL}_1 T]$ is

$$\theta + \eta = t, \quad \text{defined by} \quad W(t)(a) = \theta_a \circ \left[(\eta_{a^*})^*\right]^{-1}, \forall a \in T.$$

6. Spherical structures and canonical orientations

Call $X \in \mathbb{EF}$ *p_1-orientable* if its center admits a modular structure, which we also call a p_1-orientation on X. For a p_1-orientable X, we now describe a *canonical* p_1-orientation with optimal properties. It is the unique splitting of (5.8) in which the lifted $\pi_1 G$ centralizes T, as we shall explain below. Denoting the resulting $\mathrm{SO}^{p_1}(3)$-theory by $\mathscr{T}^1_X$,

- (i) Line operators in $\mathscr{T}^1_X$ do not require Spin structures;
- (ii) Interfaces between $\mathscr{T}^1_X$, $\mathscr{T}^1_Y$ are $\mathrm{SO}(3)$-invariant; in particular, a non-zero morphism between simple objects forces their central charges to agree mod 24;
- (iii) When $X = F$ is a fusion category and $Y = \mathbb{V}^{\otimes}$, the central charge of $\mathscr{T}^1_F$ vanishes mod 24, and $\mathscr{T}^1_F$, along with all its boundary theories, are defined on oriented manifolds.

General p_1-orientations arise from the canonical one by shearing (5.8) by elements $z \in \mathrm{GL}_1 T$ of order 2. Thus oriented, we denote the theory by $\mathscr{T}^z_X$. This need not satisfy (i–iii): in fact, part (i) characterizes the canonical p_1-orientation. For a fusion category $X = F$, the element z also determines the central charge of $\mathscr{T}^z_F$ mod 24 (0 or 12) and (in the former case) which boundary theories are $\mathrm{SO}(3)$-invariant.

(6.1) Main results. We now state our classification of orientation structures on TQFTs and compatible structures on boundaries and interfaces; some of the details must await clarification in the proof. Recall our notation $\mathbb{V}^{br\otimes}_{\mathbb{Z}/2}(\zeta)$ for the four braided versions of $\mathbb{Z}/2$-graded vector spaces.

Theorem 6. *Let $T' := \mathscr{T}_X(S^1_{nb})$ be the category for the circle with non-bounding 3-framing.*

- *(i) The $\mathrm{SO}^{p_1}(3)$-action on (the full symmetric monoidal sub-)groupoid of p_1-orientable objects in $\iota_{>0}\mathbb{EF}$ has a preferred trivialization. The canonical p_1-orientation is the constant invariance datum.*
- *(ii) This preferred invariance data for X defines a braided tensor structure on $T \oplus T'$ and factorizations*

$$GA_X = \mathrm{Aut}_{\mathbb{EF}}(X) \times G, \qquad \mathscr{T}_X(S^1_b) \oplus \mathscr{T}_X(S^1_{nb}) \xrightarrow[\text{braided}]{\sim} \mathscr{T}_X(S^1_b) \boxtimes \mathbb{V}^{br\otimes}_{\mathbb{Z}/2}(1),$$

in which $\pi_1 G = \mathbb{Z}/2$ generates the second factor $\mathbb{V}^{br\otimes}_{\mathbb{Z}/2}(1)$.

(iii) A morphism $M \in \mathrm{Hom}_{\mathbb{EF}}(X;Y)$ between canonically oriented objects admits compatible SO(3)-*invariance data.*

(iv) When $F \in \mathbb{F}$ is a fusion category and $z \in GL_1T$ with $z^2 = \mathbf{1}$, the central charge of $\mathscr{T}_F^z$ is 0 *or* 12 (mod 24)*, according to whether $\mathbb{V} \oplus \mathbb{V}z \equiv \mathbb{V}^{br\otimes}_{\mathbb{Z}/2}(1)$ or $\mathbb{V}^{br\otimes}_{\mathbb{Z}/2}(-1)$ in T.*

(v) An F-module M defines an SO(3)*-invariant boundary theory for $\mathscr{T}_F^z$ if and only if z maps to $\mathbf{1} \in \mathrm{End}_F(M)$. (This is only possible if $c(\mathscr{T}_F^z) = 0 \bmod 24$.)*

6.2 Remark. The map $Z(F) \to \mathrm{End}_F(M)$ is well-defined, because $Z(F) \cong Z\left(\mathrm{End}_F(M)\right)$. If M is simple, its invariance data is unique up to scale, and is equivalent to a Frobenius structure on an algebra object in F having M as its category of modules.

When z maps to $\mathbf{1} \in F$, the resulting orientation on F and its regular boundary gives rise to a *spherical structure*; see for instance [DSPS] for a discussion. More precisely,

6.3 Theorem. *Let F be a simple fusion category. The following are equivalent:*

- *A spherical structure on F;*
- *An* SO(3)*-invariance structure on F plus a compatible one on its regular module, up to scale.*

This meshes well with the main theorem of [FT1]: slightly loosely put, a simple fusion category is equivalent to a pair consisting of a simple TQFT $\mathscr{T}_F$ and a non-zero boundary condition.

Proof of Theorem 6.i, ii. This requires an elaboration of the arguments for Theorem 5; we develop it over the next paragraphs.

(6.4) Braided tensor structure on $T \oplus T'$. Elements of $[\mathrm{Aut}^{br} T]$ represent isomorphism classes of invertible T-modules [DN]. The Serre functor S_X corresponds to the T-module $T' = \mathscr{T}_X(S^1_{nb})$. The homomorphism $\Omega G \to \mathrm{Aut}_{\mathbb{EF}}(X)$ can be factored through the base $\mathbb{Z}/2$, after decoupling the Pontrjagin layers, and this defines a $\mathbb{Z}/2$-*crossed braided structure* [ENO] on the category

$$T \oplus T' = \mathscr{T}_X(S^1_b) \oplus \mathscr{T}_X(S^1_{nb}). \tag{6.5}$$

The E_2 group ΩGA_X of the previous section is a $K(\mathbb{Z}/4;2)$-bundle over $GL_1(T \oplus T')$ pulled back from the grading $\Omega GA_X \to \mathbb{Z}/2$. For instance, when $\pi_1 G$ injects into $[\mathrm{Aut}^{br} T]$, the factor T' is not isomorphic to T as a module and contains no invertible objects; we only see $GL_1 T$.

When $T' \cong T$, splitting the extension (5.8) lifts the homomorphism $\Omega G \to \mathrm{Aut}_{\mathbb{EF}}(X)$ to its top layer $B^2\mathbb{C}^\times$. The result deloops in four possible ways, matching the choices of $\mu(X)^{1/4}$ in splitting (5.9), and gives four promotions of (6.5) to a $\mathbb{Z}/2$-graded braided category [DN].[11] The possible outcomes are isomorphic to

$$T \oplus T' \cong T \boxtimes \mathbb{V}^{br\otimes}_{\mathbb{Z}/2}(\zeta). \tag{6.6}$$

A choice of splitting of (5.5) determines a specific value of ζ: the group $\pi_1 G = \mathbb{Z}/2$ must lift to a copy of $\mathbb{V}^{br\otimes}_{\mathbb{Z}/2}(1) \subset T \oplus T'$. We can arrange for this to be the second factor in (6.6) by shearing any splitting of (5.5) by a suitable order-2 element in $GL_1 T$, unique up to isomorphism.

6.7 Definition. The *canonical* modular structure on T and p_1-orientation of X are defined by splitting $GA_X \subset T \oplus T'$ by the generator κ of $\mathbb{Z}/2$ in the factorization (6.6). By $\mathscr{T}_X^z$, we denote the p_1-orientated TQFT defined by shearing the canonical splitting by a 2-torsion element $z \in GL_1 T$.

Necessarily, $\zeta = 1$ for the canonical orientation; for other choices of z, ζ must cancel the braiding on $\mathbb{V} \oplus \mathbb{V} \cdot z \subset T$.

[11] The automorphism group of Id_T is $B^2\mathbb{C}^\times$, and H-graded braided categories with an abelian group H and identity component T are classified by E_2 homomorphisms $H \to \mathrm{Aut}(\mathrm{Id}_T)$.

(6.8) Reformulation: the ribbon on T'. The framing-preserving circle action on S^1_{nb} defines an automorphism θ' of the identity of T'. This θ' is a quadratic refinement of the braiding, in that

$$\theta'(x \otimes y \otimes z)\theta'(x \otimes z)^{-1}\theta'(y \otimes z)^{-1}\theta'(z) = \beta_{y,x} \circ \beta_{x,y} \otimes \mathrm{Id}_z, \quad \forall x, y \in T, z \in T'$$

A trivialization of S_X is a choice of T-module isomorphism $T \cong T'$, identifying the two Spin circles, and is effected by an invertible object $b \in T'$. This transports θ' to a balancing on T,

$$b^*\theta' : a \mapsto \theta'(a \otimes b) \circ \left(\mathrm{Id}_a \otimes \theta'(b)^{-1}\right).$$

The $\theta'(b)$-correction is to ensure that $b^*\theta'(\mathbf{1}) = 1$. Comparing with the definition of ρ (end of the proof of Theorem 5), this is seen to be a ribbon iff $b^2 \cong \mathbf{1}$. Changing b by a 2-torsion element $t \in T$ changes the double-braiding action of $W(b)$ on T by $W(t)$. Non-degeneracy of β on T ensures that *exactly one* choice κ of b is central in $T \oplus T'$. The resulting canonical structure splits $\pi_1 G$ in $GA_X \subset T \oplus T'$. Any other splitting differs from it by some order-two element $z \in GL_1 T$. □

Proof of Theorem 6.iv. We know from Theorem 4 that $\mu(F) = 1$. Noting that $\mathbb{V}^{br\otimes}_{\mathbb{Z}/2}(\pm i)$ cannot appear in the center of a fusion category F (because we need a central functor to one of the fusion subcategories $\mathbb{V}^{\otimes}$ or $\mathbb{V} \oplus \mathbb{V} \cdot z$), the statement now follows from the observation following Definition 6.7: we will pick one of the values $\mu(F)^{1/4} = \pm 1$, matching the braiding on $\mathbb{V} \oplus \mathbb{V} \cdot z$. □

Proof of Theorem 6.iii, v. In part (iii), assume that X, Y are simple, and use the folding trick to pass to the fusion category $F := X \boxtimes Y^{\vee}$, for which M becomes a module: we are reduced to Part (v).

View M as an object in $\mathrm{Hom}(\mathbb{V}^{\otimes}; F)$. The latter is the fiber of a bundle over $B\mathrm{SO}^{p_1}(2)$, because of the invariance of F. The 2-skeleton $S^2 \to B\mathrm{SO}^{p_1}(2)$ gives an integral cohomology isomorphism through H^4. The obstruction to a section through M over $B\mathrm{SO}^{p_1}(2)$ is then detected over S^2, where we meet the action of $\pi_1\Omega S^2 = \mathbb{Z}$ via the Serre functor S_M relative to F.[12] If S_M is trivializable in a way compatible with the isomorphism $S_F \cong \mathrm{Id}_F$, the categorical cutoff of M precludes a p_1-dependence in F, and forces the $\mathrm{SO}(3)$-invariance of F and M.

To find a trivialization, we may assume that F and M are simple. Replacing F by the isomorphic object $\mathrm{End}_F(M) \in \mathbb{F}$ reduces us to the case of the regular module $M = {}_F F$. The Serre functor S_F is identifiable with the double dual $**$ on the fusion category F (see [DSPS]). This also identifies the relative Serre functor S_M with the double dual. A tensor isomorphism $\mathrm{Id}_F \cong **$ would give the sought-after compatible trivialization of S_M.

Now, a trivialization of $S_F \in \mathrm{Aut}_{\mathbb{F}}(F)$ is a tensor isomorphism, for some fixed $u \in F$,

$$\mathrm{Id}_F \cong \{x \mapsto u \otimes x^{**} \otimes u^{-1}\}, \tag{6.9}$$

and a compatible trivialization of S_M is a functorial isomorphism $\{m \xrightarrow{\sim} u \otimes m^{**}\}$ on the F-module M. This exists iff $u = \mathbf{1} \in F$, in which case the trivializations form a torsor over the automorphisms $\mathbb{C}^{\times}$ of the identity functor of M (as F-module). We show first that $u = \mathbf{1}$ for the canonical orientation. To do so, we shift the problem to the center $Z(F)$ by "squaring" the pair (F, M) to the category $F \boxtimes F^{op}$ and its regular module. This leads to the element $u \boxtimes u^{op}$ in the analogue of (6.9). A compatible boundary orientation exists iff $u \boxtimes u^{op} = \mathbf{1}$, which in turn happens iff $u = \mathbf{1}$.

We find the compatible orientation by replacing the pair $(F \boxtimes F^{op}, F \boxtimes F)$ in $\mathbb{F}$ with the isomorphic pair $(Z(F), F)$. Choosing $\mathbf{1} \in F$ as generating object over $Z(F)$, the module F gets identified

[12]Here, there is no possible extension-class obstruction, as in the sequence (5.8).

with the category of (right) module objects internal to $Z(F)$ over the commutative algebra object $W := \iota^*(\mathbf{1})$, where $\iota : Z(F) \to F$ is the natural map. The functor Id_F corresponds to the W-W-bimodule W, and S_F to W^{**}. On $Z(F)$, the bulk Serre functor is identified with the identity by the factorization (6.6): κ identifies the bimodule T' with the identity bimodule T. Relative orientability of F then amounts to the agreement of the Morita equivalence $W^{**} \equiv W$, mediated by W^* in $Z(F)$ (the bottom edge in Figure 3), with the identification $W \cong W^{**}$ induced by the canonical orientation of $Z(F)$ (the right edge). Left-dualizing once, this datum is equivalent to an isomorphism of W-W bimodules

$$W \cong (W^* = {}^*W)$$

with the natural left-and-right actions of W on the two right objects, and their identification via the canonical $** = \mathrm{Id}_{Z(F)}$.

The obvious isomorphisms[13] out of W,

$$W \cdot \mathbf{1}^* \to W^*, \qquad {}^*\mathbf{1} \cdot W \to {}^*W,$$

are adjoint to the composition $W \otimes W \to W \to \mathbf{1}$, differing in the order of the factors, but with the (commutative) multiplication as the first step. They are identified by the Drinfeld isomorphism in $Z(F)$. This differs from the orientation-induced isomorphism by the balancing $\theta = \kappa^*\theta'$, pulled back from θ' on T' as in §6.8. We claim that $\theta \equiv 1$ on W. The geometric proof is contained in Figure 2: $\kappa \cdot W \in T'$ is the output of the the boundary F in the annulus, with the strict rotation-invariant framing.

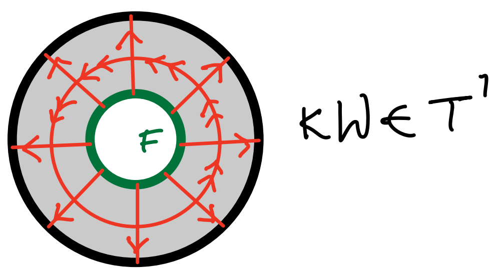

Figure 2: We have $\theta' \equiv 1$ on $\kappa \cdot W$ because of strict rotation-invariance

The case of a general bulk orientation z follows by noting that the isomorphism defining our orientation structure,

$$z\otimes : \mathrm{Hom}_{F\text{-}F}(F; F) \to \mathrm{Hom}_{F\text{-}F}(F; F^\vee),$$

changes the identification $W \cong W^{**}$ by z-conjugation. We then need to identify the W-W bimodules W and zW. This can be done iff $\iota(z) = \mathbf{1} \in F$. □

6.10 Remark. Figure 3 illustrates the key check required for relative p_1-orientability of M. The bulk Serre functor S_F (blue) can terminate on the boundary M as a framing twist S_M. An isomorphisms between the two equivalences around the square defines a termination (black dot) of the bulk end (red) of S_F on the boundary M, ending (and therefore trivializing) S_M.

Proof of Theorem 4. Orientability of F and its regular module allows us to draw the two isomorphic pictures in Figure 4 for the left and right trace of an object $x \in F$ exhibiting the sphericality condition [EGNO, Def. 4.7.14]. The theory $\mathscr{T}_F$ is applied to the solid ball, with the regular boundary, and a self-defect boundary loop labeled by x.

[13] Isomorphy holds because they are non-zero maps of invertible bi-modules.

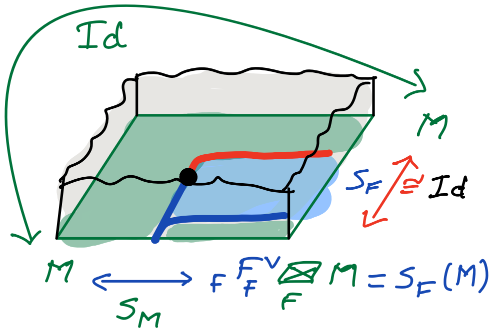

Figure 3: Trivializing the boundary Serre functor S_M

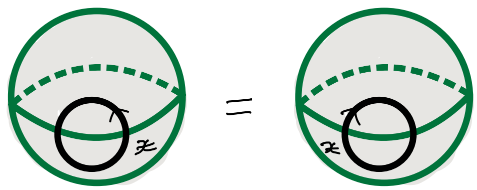

Figure 4: The unlabeled arm of the loop can be read as either ${}^*x$ or x^*.

Conversely, as shown by Turaev [Tu] and Müger [M], a spherical structure on F leads to a modular structure on $Z(F)$. The canonical orientation then gives F a preferred piovotal structure. Any other one differs from it by a tensor automorphisms of Id_F, which is the double-braiding $W(z)$ of (5.11) with a unique invertible $z \in \iota^{-1}(\mathbf{1}) \in Z(F)$. When relating spherical structures, this z must have order 2, so the original spherical structure must be among the ones thus obtained. □

(6.11) Four variants of $\mathbb{Z}/2$ gauge theory. The center Z of $\mathbb{V}^{\otimes}_{\mathbb{Z}/2}$ (with its untwisted associator) has four simples $\{\mathbf{1}, e, m, em\}$, where m is the generator of $\mathbb{Z}/2$ and e is its sign representation. The framed theory has two simple boundary conditions, Neumann and Dirichlet, and four simple line operators from Z. Line operators with $\theta = -1$ will always need normal framings (here, mod 2, as it happens). The four possible ribbons are listed in Table 1.

	$\mathbf{1}$	m	e	em
θ^1	1	1	1	-1
θ^m	1	1	-1	1
θ^e	1	-1	1	1
θ^{em}	1	-1	-1	-1

Table 1: Ribbons on Z

The four $\mathbb{Z}/2$-gauge theories are defined by counting classical fields (summing in a categorical sense, when appropriate). The first three are genuinely oriented theories, while the fourth one has a sign anomaly. Here are their explicit descriptions:

(i) θ^1 is the usual bundle-counting theory, defined even for unoriented manifolds. Neither boundary condition requires a Spin structure, nor do any of the line operators.

(ii) θ^m is the twisted version where the classical fields are Spin structures, instead of double covers. The Dirichlet boundary, but not the Neumann one, requires a boundary Spin structure.

The line operators e and em require Spin structures along their support: we need a base Spin structure to compare with the Spin field in order to define the monodromy of a Wilson loop.

(iii) θ^e counts bundles with weight $w_2 \cup w(m) \in K(\{\pm 1\};3)$, with the class $w(m) \in H^1$ of the bundle. The Neumann condition (but not the Dirichlet one) now requires a Spin structure, to trivialize this twist. The 't Hooft loop m needs a longitudinal Spin structure (as does em): $w(m)$ is not defined on the loop, so w_2 must be trivialized there.

(iv) θ^{em} is the Spin-counting theory *relative to the* p_1*-structure* τ, $\delta\tau = p_1$: a Spin structure is a cochain $w(m)$ trivializing w_2, so $w_2 \cup w(m)$ trivializes p_1 mod 2. The difference $(\tau - w_2 \cup w(m)) \in K(\{\pm 1\};3)$ is our counting twist. This theory has no $SO(2)$-invariant boundary conditions; its central charge is 12 mod 24, since a unit shift in τ changes the sign of the top invariants.

(6.12) Non-canonical orientations on F. These four versions of this gauge theory can be grafted onto a theory $\mathscr{T}_F$ to effect a change from the canonical orientation. Let $z \in Z(F)$ have order 2. Roughly, we tensor F over $\mathbb{V} \oplus \mathbb{V} \cdot z$ with the four gauge theories, but the detailed story depends on whether or not z maps to $\mathbf{1} \in F$.

When z maps to $\mathbf{1} \in F$. This corresponds to θ^e above. Half-braiding with z defines a grading $F = F_0 \oplus F_1$ on F, which realizes F as the $\mathbb{Z}/2$-gauging of F_0. We can build $\mathscr{T}_F$ by gauging F_0 in the variant structure θ^e, adding the w_2-weighting when counting bundles.

When z does not map to $\mathbf{1} \in F$. This orientation twist has two possible outcomes, matching θ^m or θ^{em}, depending on whether $\langle \mathbf{1}, z\rangle$ is $\mathbb{V}^{br}_{\mathbb{Z}/2}(1)$ or $\mathbb{V}^{br}_{\mathbb{Z}/2}(-1) = S\mathbb{V}$. (The $\zeta = \pm i$ braidings cannot occur, because $\langle \mathbf{1}, z\rangle$ acts centrally on the same subcategory of F.) We use z to 'couple the category F to the Spin structure' on the manifold where $\mathscr{T}_F$ is to be evaluated. The precise TQFT picture realizes $\mathscr{T}_F$ as a boundary of the 4D gerbe theory defined by $\langle \mathbf{1}, z\rangle$, and builds a sandwich by placing the Spin-counting theory on the opposite side.

7. Complex p_1-structures and central charge

We now introduce an enhanced tangential structure, a *complex* p_1*-structure*, capable of seeing the lift of $\underline{c}$ to a complex number.

7.1 Definition. A $\mathbb{C}p_1$-structure on a real vector bundle is a trivialization of the first complexified Pontrjagin class.

The stable group $\mathrm{Spin}^{\mathbb{C}p_1}$ is a $\Sigma^2\mathbb{C}$-extension of Spin. Restricting to dimensionally-fixed groups, such as $\mathrm{Spin}^{\mathbb{C}p_1}(2,3)$, leads to $\mathbb{C}p_1$-tangential structures on manifolds. The relevance of this definition is captured by the following; the fist statement is self-explanatory, the second may need the review in §7.3 at the end of this section.

Theorem 7. *(i) A choice* $\lambda(X)$ *of* $\log \mu(X)$ *extends* $\mathscr{T}_X$ *to* $\mathrm{Spin}^{\mathbb{C}p_1}(3)$*-manifolds, in such a way that a shift in* $\mathbb{C}p_1$*-structure by* $z \in \mathbb{C}$ *scales the manifold invariants by* $\exp(z\lambda(X))$.

(ii) A chiral Spin *CFT with central charge* $c > 0$ *can be a boundary theory for the* $\mathrm{Spin}^{\mathbb{C}p_1}(3)$ *theory* $\mathscr{T}_X$ *only if* $\lambda(X) = 2\pi i c/6$.

We first describe the relevant $\mathbb{C}p_1$ bordism groups.

7.2 Proposition. *(The reader should mind the conventional shifts in MT spectra.)*

(i) $\mathbb{S} \to M\mathrm{Spin}^{\mathbb{C}p_1}$ *induces isomorphisms on* $\pi_{0,1,2}$ *and the inclusion of* $\mathbb{Z}/24$ *in* $\mathbb{C}/48\mathbb{Z}$ *on* π_3.

(ii) $MT\mathrm{Spin}^{\mathbb{C}p_1}(3) \to \Omega^3 M\mathrm{Spin}^{\mathbb{C}p_1}$ *induces isomorphisms on* $\pi_{<0}$ *and a splitting*

$$\pi_0 MT\mathrm{Spin}^{\mathbb{C}p_1}(3) = \pi_3 M\mathrm{Spin}^{\mathbb{C}p_1} \oplus \pi_0 MT\mathrm{Spin}(3) \cong \mathbb{C}/48\mathbb{Z} \oplus \mathbb{Z}/2.$$

Under $\mathbb{S}^{-3} \to MT\mathrm{Spin}^{\mathbb{C}p_1}(3)$, *the generator of* π_0 *maps to* $(2,1)$.

(iii) $MT\mathrm{Spin}^{\mathbb{C}p_1}(2) \to \Omega^2 M\mathrm{Spin}^{\mathbb{C}p_1}$ *induces isomorphisms on* $\pi_{<0}$ *and a splitting*

$$\pi_1 MT\mathrm{Spin}^{\mathbb{C}p_1}(2) = \pi_3 M\mathrm{Spin}^{\mathbb{C}p_1} \oplus \pi_1 MT\mathrm{Spin}(2) \cong \mathbb{C}/48\mathbb{Z} \oplus \mathbb{Z}/4,$$

while $\pi_0 MT\mathrm{Spin}^{\mathbb{C}p_1}(2) = \mathbb{Z} \oplus \mathbb{Z}/2$.

Under $\mathbb{S}^{-2} \to MT\mathrm{Spin}(2)$, η^2 *maps to* $(0,1)$ *and the generator of* π_3^s *maps to* $(2,1)$.

In all cases, an integral unit shift of p_1*-structure represents* $1 \in \mathbb{C}/48\mathbb{Z}$.

Proof. (i) The Atiyah-Hirzebruch spectral sequence computing $\mathbb{I}^*_{\mathbb{C}^\times}(MT\mathrm{Spin})$ starts with

$$E_2^{p,q} = H^p\left(B\mathrm{Spin}; \pi_{-q}\mathbb{I}_{\mathbb{C}^\times}\right).$$

The vanishing of Ω_3^{Spin} ensures the injectivity of the only differential originating at $E_2^{0,3}$,

$$d_4 : (\pi_3^s)^\vee \to H^4\left(B\mathrm{Spin}; \mathbb{C}^\times\right) = \mathbb{C}/\mathbb{Z} \cdot \frac{p_1}{2}.$$

This d_4 then represents a generator of the group $\mathrm{Ext}^1(\mathbb{Z}/24; \mathbb{Z})$. The extension is the group

$$\mathbb{I}^0_{\mathbb{Z}}(MT\mathrm{Spin}) = \mathbb{Z} \cdot \frac{p_1}{48},$$

surjecting onto $(\pi_3^s)^\vee$. This is Pontrjagin dual to the inclusion in the Proposition.

(ii) The same calculation, now minding that $\ker d_4 = \pi_0 MT\mathrm{Spin}(3) = \mathbb{Z}/2$, gives the abstract isomorphism. The asserted splitting follows, because the theory ν of §B.5 detects the kernel of d_4.

(iii) Similar to (ii). □

Proof of Theorem 7.i. An extension of $\mathscr{T}_X$ to $\mathrm{Spin}^{\mathbb{C}p_1}(3)$-manifolds is equivalent to a trivialization of the respective group action on X. Through dimension 3, $\mathrm{Spin}^{\mathbb{C}p_1}(3) = \Sigma^2 H\mathbb{C}/\mathbb{Z}$, and it acts via the Bockstein

$$\Sigma^2 H\mathbb{C}^\times \xrightarrow{B} \mathrm{Spin}(3) = \Sigma^3 H\mathbb{Z} \xrightarrow{\mu(X)} \Sigma^3 H\mathbb{C}/\mathbb{Z}.$$

A lift $\lambda : \mathbb{Z} \to \mathbb{C}$ of μ trivializes the composite homomorphism $\mu \circ B$, with the stated transformation law under shift of structure. □

(7.3) Coupling to 2*-dimensional boundary theories.* Chiral CFTs, if not topological themselves, have a non-zero *central charge* $c \in \mathbb{C}$, manifested as a $\mathbb{C}^\times$-central extension of the diffeomorphism group $\mathrm{Diff}(S^1)$ of the circle. They may also require Spin structures on circles and surfaces. Consistent coupling to a TQFT as a conformal boundary theory—meeting Segal's *modular functor* axioms [S]—requires a matching of tangential structures. For Spin structures, the meaning is clear, so we focus on the central charge. Here is the main result.

Proof of Theorem 7.ii. A central extension of $\mathrm{Diff}(S^1)$ defines a class in its smooth group cohomology with $\mathbb{C}^\times$ coefficients, which in this case is an extension of the cohomology of the Lie algebra $\mathfrak{diff}(S^1)$ by the de Rham cohomology of the classifying space,[14]

$$\mathbb{C}^\times = H^2\left(B\mathrm{Diff}(S^1); \mathbb{C}^\times\right) \to H^2\left(B\mathrm{Diff}(S^1); \mathscr{O}^\times\right) \to H^2\left(\mathfrak{diff}(S^1); \mathbb{C}^\times\right) = \mathbb{C}.$$

[14] This is a simple instance of the *van Est* spectral sequence.

Restriction to $\mathrm{SL}(2;\mathbb{R})$ splits the sequence, because the Lie algebra cohomology of $\mathfrak{sl}(2;\mathbb{R})$ vanishes. The $\mathbb{C}^\times$ component is the *conformal weight* of a projective representation of $\mathrm{Diff}(S^1)$: it measures the failure of the rigid rotation subgroup to lift to the central extension through the Lie algebra splitting enforced by $\mathfrak{sl}(2;\mathbb{R})$. The $\mathbb{C}$ component is the *central charge*.

The distinguished value 6 of the central charge has the following topological meaning. Over the classifying space of Spin(3)-bundles *with connection*, $p_1/4$ has a *differential cohomology* refinement $\check{p}_1/4 \in \check{H}^4(B_\nabla\mathrm{Spin}(3);\mathbb{Z})$. (The Chern-Simons 3-form may be used to that effect.) Integrating fiberwise in the universal (3-framed) circle bundle (in the bounding 3-framing) leads to a left-invariant class in $\check{H}^3$ of the base $B_\nabla\mathrm{Diff}(S^1_b)$. This is the $c = 6$ central extension of $\mathrm{Diff}(S^1)$ by $\mathbb{C}^\times$; we denote it[15] $\mathrm{Diff}^{p_1/4}(S^1)$.

This last group acts *projectively* on the Hilbert space sectors of the CFT, but its Lie algebra (the *Virasoro algebra*), and its universal cover, act linearly. The result is a *topological* action of $\mathrm{Diff}^{p_1/4}(S^1)$ on the category $\mathscr{C}f$ of these Virasoro representations. (See [T] for topological group actions on categories.) This topological action only sees a small group: the $p_1/4$-extension of $B\mathbb{Z}$ by $B\mathbb{Z}$, corresponding to that of the rotation group S^1 by the central S^1. (The extension has been trivialized earlier, using $\mathfrak{sl}(2;\mathbb{R})$.) Detected by this action are $c \mod 6$, and the conformal weights.

The group $B\mathbb{Z} \times B\mathbb{Z}$ also acts on the semi-simple category $\mathscr{T}_X(S^1)$ associated to an object $X \in \mathrm{E}S\mathbb{F}$; the central $B\mathbb{Z}$ acts by $\mu(X)$, while the rotation $B\mathbb{Z}$ acts by a Spin version of the ribbon automorphism of objects.

Segal's *modular functor*, matching a rational 2-dimensional CFT to a 3D TQFT $\mathscr{T}_X$, includes a functor $\mathscr{T}_X(S^1) \to \mathscr{C}f$. This functor must be invariant under the topological $\mathrm{Diff}(S^1)$-actions on the two categories. This requires matching the ribbon values with conformal weights, and $\mu(X)$ with $\exp(2\pi i c/6)$.

Using, in the same argument, general multiples of p_1, leads to powers $\mathrm{Diff}^{\,kp_1}(S^1)$ of our central extension, and enforces the equality $\lambda(X) = c \mod 24k$. □

7.4 Remark. It is important to note that we are not free to change the topological action of $\mathrm{Diff}(S^1)$ on $\mathscr{C}f$ for theories with $c \neq 0$. This is because the functor we construct requires invariance under the group, and the first step, the descent from a genuine to a topological action on $\mathscr{C}f$, uses the Virasoro action on the Hilbert spaces. This fixes the projective $\mathrm{Diff}^{p_1/4}(S^1)$-action on the same. The source objects must carry topological charges cancelling the (now flat) projective cocycles of the representation.

7.5 Remark. The central charge match in Theorem 7.ii is not sufficient: the unstable summand $\mathbb{Z}/4$ in Proposition 7.2 also requires matching. We do not know an instance where this problem appears naturally, although of course we can tensor the TQFT with an unstable invertible factor to create a mismatch.

8. Projective symmetric tensor structures

Our construction of $\mathrm{E}S\mathbb{F}$ took care to represent the homotopy-abelianized symmetric group $\mathbb{S}^{-1}_{\geq 0}$ on the tensor structure. For bosonic theories in $\mathrm{E}\mathbb{F}$, which forgo super-vector spaces and Clifford algebras, we could quotient out η in the symmetry. The effect of our constructions on the 4-categories $\mathbb{B}$ and $S\mathbb{B}$ was to quotient out the Witt group of invertible objects, by adding isomorphisms with the unit objects $\mathbb{F}, S\mathbb{F}$. This quotienting in the *target* is complementary to quotienting in $\mathbb{S}$, as we will now see by introducing *projectivity* in the symmetric monoidal structure. This is

[15] There are two versions of the group, for the two Spin circles; for the bounding Spin circle, the rotation must be doubled, but this only concerns the normalization of the conformal weights.

distinct from the notion of a projective, or anomalous TQFT, as discussed in the introduction: we will produce symmetric monoidal functors between projectively symmetric monoidal categories (with the same projective cocycle).

Specifically, we will use the projectively-symmetric monoidal bordism 3-category $\mathrm{Bord}^{\sigma/2}$ of oriented ≤ 3-manifolds with *half-signature structure* as a source of our TQFTs. This leads to unique fully local, half-signature-structured TQFTs from modular tensor categories, a slight amendment of Turaev's signature structures that removes the sign ambiguity.

Theorem 8. *The six TQFTs defined from modular tensor category T on the 3-category of $\mathrm{SO}^{p_1}(3)$-manifolds and valued in $\mathbb{EF}$ are linearizations of a single symmetric monoidal functor $\mathrm{Bord}^{\sigma/2} \to \mathbb{EF}/\langle U\rangle$ valued in the projectively-symmetric tensor category obtained from $\mathbb{EF}$ by collapsing the group $\langle U\rangle$ of units.*

The last category should need no explanation: the units $\langle U\rangle$ were added to $\mathrm{E}'\mathbb{F}$, yielding $\mathbb{EF}$, for the express purpose of resolving the projective anomaly in the symmetric tensor structure: leaving them out gives the projectively symmetric structure. Signature structures would leave a sign ambiguity in defining the projective theory. We first illustrate the method in a toy example, forcing an isomorphism between the odd and even Clifford algebras in $S\mathbb{L}$.

(8.1) Warm-up: Clifford algebras. The category of $\mathbb{Z}/2$-graded Cliff(1)-modules is generated by the Spin module Γ. This is invertible, but not free over $S\mathbb{V}$: for instance, the tensor action of the odd line is $\mathbb{V}$-linearly isomorphic to the identity, which is not the case in $S\mathbb{V}$. The invertible *Arf theory* on Spin(2)-manifolds takes the value $\langle\Gamma\rangle$ on a point and detects the non-trivial character $\pi_2\mathbb{S} \to \mathbb{C}^\times$.

8.2 Theorem. *We can introduce an equivalence* $\mathrm{Cliff}(1) \equiv \mathbb{C}$ *in the 2-category $S\mathbb{L}$ while keeping the tensor structure projectively-symmetric tensorial.*

Attempting to replicate the construction of $\mathbb{EF}$ will encounter an irremovable (higher) projective cocycle for the symmetry, valued in a homotopy 3-type built as a quotient of the sphere.

Proof. We attempt to define a symmetric tensor structure on the direct sum category[16] $S\mathbb{V} \oplus \langle\Gamma\rangle$, the analogue of $\mathbb{EF}$, so that

$$\Gamma \otimes \Gamma^\vee = \mathrm{Cliff}(1);$$

in other words, we wish to tensor over Cliff(1) instead of $\mathbb{C}$, as we did in §1. Were we to succeed, Γ would define the desired Morita equivalence $\mathrm{Cliff}(1) \equiv \mathbb{C}$.

However, the Arf theory defined by $\langle\Gamma\rangle$ detects $\eta^2 \in \pi_2^s$ as its top invariant, and the resulting map $\mathbb{S} \to \Sigma^2\mathbb{I}_{\mathbb{C}^\times}$ *does not* factor through $H\mathbb{Z}$. Removing this η^2 obstruction requires restricting the symmetry to a spectrum $\mathbb{S}'$, fiber of

$$\mathbb{S}' \rightarrowtail \mathbb{S} \twoheadrightarrow A$$

where A has homotopy groups $\pi_0 = \mathbb{Z}/4$, $\pi_1 = \langle\eta\rangle$, $\pi_2 = \langle\eta^2\rangle$, quotients of the respective π_*^s. The spectrum $\Omega\mathbb{S}'$ is the portion of the symmetric group $\mathfrak{S}_\infty$ that is compatible with this novel tensor structure on $S\mathbb{V} \oplus \langle\Gamma\rangle$; it is an extension of $\mathfrak{S}_\infty$ by a 2-group, and the standard co-extension of its classifying space to the set $\mathbb{Z}$ of components is obstructed mod 4. □

8.3 Remark. This restricted symmetric tensor structure may be reinterpreted by enhancing our bordism 2-category to points with *A-structure:* a $\mathbb{Z}/4$-grading and 2-group of automorphisms ΩA, with multiplication reflecting that of the spectrum A. The internal structure then 'absorbs' the quotient A of $\mathbb{S}$, and the residual part of the symmetry is precisely $\mathbb{S}'$.

[16]A special feature of this toy example is that $\Gamma^\vee \cong \Gamma$, but this is unrelated to the obstructions.

(8.4) Twisted tangential structures. We normally define a tangential structure (Appendix B) as a lift of the classifying map of the tangent bundle $M \to B\mathrm{O}(n)$ of n-manifolds to a specified structure space $S \to B\mathrm{O}(n)$. (The fibers of S are the *topological background fields*.) An example of such S is the homotopy fiber of a map $B\mathrm{O}(n) \to B$; here, B could be a classifying spectrum for a cohomology theory, such as $\Sigma^4 H\mathbb{Z}$, which we used for p_1-structures.

A variant involves *twisted* cohomologies, which arise as *sections of bundles* of spectra over $B\mathrm{O}(n)$, rather than maps to spectra. We specialize to $\mathrm{SO}(n)$, and twist by the formal negative of the standard n-dimensional representation R. Our cohomology theory will be the integral (Anderson) dual $\mathbb{I}_{\mathbb{Z}}$ of the stable sphere. The twisted groups $\mathbb{I}_{\mathbb{Z}}^{*-R}(B\mathrm{SO}(n))$ are precisely the $\mathbb{I}_{\mathbb{Z}}^{*}$-cohomologies of the spectrum $MT\mathrm{SO}(n)$. For $n = 3$, these are

$$\mathbb{I}_{\mathbb{Z}}^{-R} = \mathbb{Z}, \quad \mathbb{I}_{\mathbb{Z}}^{1-R} = \mathbb{I}_{\mathbb{Z}}^{2-R} = \mathbb{I}_{\mathbb{Z}}^{3-R} = 0, \quad \mathbb{I}_{\mathbb{Z}}^{4-R} = \frac{1}{6}\mathbb{Z} \cdot p_1, \tag{8.5}$$

showing that the signature $\sigma := p_1/3$ and its half $\sigma/2 = p_1/6$ are characteristic classes in this twisted cohomology theory. We can compare these with the same groups for $B\mathrm{SO}^{p_1}(3)$,

$$\mathbb{I}_{\mathbb{Z}}^{-R} = \mathbb{Z}, \quad \mathbb{I}_{\mathbb{Z}}^{1-R} = \mathbb{I}_{\mathbb{Z}}^{2-R} = \mathbb{I}_{\mathbb{Z}}^{3-R} = 0, \quad \mathbb{I}_{\mathbb{Z}}^{4-R} = \left(\frac{1}{6}\mathbb{Z}/\mathbb{Z}\right) \cdot p_1;$$

these match the respective dual homotopy groups $\mathbb{Z}, 0, 0, \mathbb{Z}/6$ of the co-fiber of $\mathbb{S}^{-1}/2 \xrightarrow{\eta} \mathbb{S}$, which controls the symmetry of $\mathbb{E}\mathbb{F}$. In particular, the fractional multiples of p_1 in (8.5) are non-zero on the sphere — the symmetric monoidal point.

(8.6) Signature-related structures. Because of their negative tangent twist, the classes $\sigma, \sigma/2$ can be integrated on SO(3)-manifolds without additional orientation constraints or choices.

8.7 Definition. A *signature structure* on a manifold M is a trivialization of $\int_M \sigma$. A *half-signature structure* is a trivialization of $\int_M \sigma/2$.

When M has boundaries and corners, the integral must be trivialized relative to pre-existing boundary/corner trivializations.

Now, σ or $\sigma/2$ do not admit trivializations over the sphere, the symmetric monoidal point: the class is non-zero. Nonetheless, signature structures may exist, and behave additively, on 3-manifolds with corners up to co-dimension 2, because of the vanishing of the respective bordism groups. The integral of these classes over *three*, respectively six points, can also be trivialized, because $3\sigma = p_1$ vanishes canonically. We can thus define a signature structure on triples of points, and half-signature structures on sextuplets.

(8.8) Confinement and deconfinement. In this version of signature structures, points must travel in triples, à la "quark confinement." But we can break up the trios with the same device of introducing internal structures for points, on which the symmetric group is represented.

Namely, the sphere fits in a fibration of spectra

$$\mathbb{S}'' \rightarrowtail \mathbb{S} \twoheadrightarrow Q = H\mathbb{Z}/3 \ltimes^{P_3^1} \Sigma^3 H\mathbb{Z}/3$$

with $\pi_0 Q = \pi_3 Q = \mathbb{Z}/3$, connected by the Steenrod power P_3^1. We may think of Q as a derived Heisenberg group; it represents a $\mathbb{Z}/3$-graded $\Sigma^3\mathbb{Z}/3$ torsor. Since σ is null-homotopic in $[\mathbb{S}''; \tilde{I}_{\mathbb{Z}}^4]$, signature structures certainly exist on $\mathbb{S}''$-parametrized 0-manifolds.

We can now represent the full symmetric group (the sphere) on the category obtained by an 'induction along Q', a direct image from $\mathbb{S}''$ to $\mathbb{S}$. Our points now carry a $\mathbb{Z}/3$-grading, as well as an internal $\Sigma^2 H\mathbb{Z}/3$ group of automorphisms, interacting in the way described by Q, whose π_3 is represented on $\mathbb{C}^\times$ in the top-level automorphisms of $\mathbb{EF}$ via the signature σ. Triples of points have degree 0, and behave fully symmetrically; but the symmetry on individual points is governed by a generalized Koszul sign rule, which takes the grading of points into account.

Proof of Theorem 8. This is obvious from the matching of the $\mathbb{Z}/6$ groups in the sphere and in the half-signature structure. □

A. An exotic gauge theory

The new objects in $\mathbb{EF}$ allow the construction of additional 4-dimensional TQFTs, which do not come from the fusion 2-categories classified in [CF2]. This is why we cannot quite rely on the latter to construct the one-step categorifications of $\mathbb{EF}$ and $\mathbb{ESF}$, the expected targets for 4-dimensional TQFTs. We will return to this in the follow-up paper [FST]; here, we describe the exotic bosonic 4-dimensional $\mathbb{Z}/3$ gauge theory made possible by the group of units $\langle U\rangle \subset \mathbb{EF}$ from §1.

The object $A := \mathbf{1} \oplus U^{\otimes 2} \oplus U^{\otimes 4} \in \mathbb{EF}$ has a canonical algebra structure: the vanishing group $H^4\big(B\mathbb{Z}/3;\mathbb{C}^\times\big)$ of Dijkgraaf-Witten twists precludes any variation of the (higher) associator. The 4-dimensional TQFT $\mathscr{G}_{\mathbb{Z}/3;\delta}$ that it generates[17] is a gauge theory for the group $\mathbb{Z}/3$ with the novel Dijkgraaf-Witten twist δ, valued in the group of units $\mathrm{GL}_1\,\mathbb{EF}$. The classifying space of the latter has $\pi_4 = \mathbb{C}^\times$, $\pi_1 = \mathbb{Z}/6$ and k-invariant $Sq^2 \times P_3^1$; this vanishes at this stage of delooping, and the map

$$\delta : B\mathbb{Z}/3 \to B\mathrm{GL}_1\,\mathbb{EF}$$

is the unique lift of the inclusion in the base (using the cohomology vanishing just mentioned.)

It is easy to compute $\mathscr{G}_{\mathbb{Z}/3;\delta}(S^1)$ as an object in $\mathbb{EF}$: the answer is

$$\left(\mathbf{1} \oplus U^{\otimes 2} \oplus U^{\otimes 4}\right) \boxtimes \mathbb{V}_{\mathbb{Z}/3}, \tag{A.1}$$

the original object tensored with the generator of 3-dimensional gauge theory.

We can now evaluate the reduced theory on 3-spheres with various framings. The $\mathbb{V}_{\mathbb{Z}/3}$ factor gives $1/3$, from bundle automorphisms, but the presence of U in the first factor leads to framing-dependent answers: we get 3 for framings representing multiples of 3 in the bordism group, and 0 otherwise.

This TQFT does not match the standard $\mathbb{Z}/3$ gauge theory (generated by the algebra $\mathbf{1}\oplus\mathbf{1}\oplus\mathbf{1}$, with the obvious $\mathbb{Z}/3$-multiplication), as the latter is insensitive to framing; nor can it be built from algebra objects in $\mathbb{F}$. Indeed, from (A.1), we compute the Müger center $\mathscr{T}_{\mathbb{Z}/3;\delta}(S^2)$ to be $\mathrm{Rep}(\mathbb{Z}/3)$, so any candidate must be some version of $\mathbb{Z}/3$-gauge theory. However, there are no Dijkgraaf-Witten twists that we could use, without the units U.

B. Tangential structures involving p_1

We review the p_1-tangential structures on 3-manifolds and on boundary surfaces, where TQFTs can interface with Conformal Field Theories. These results are applied to the TQFTs $\mathscr{T}_X$ in §3.

(B.1) Variations. Framing a 3-manifold defines a Spin structure. This is preserved by a *local* change of framing (one concentrated near a point), because maps $S^3 \to \mathrm{SO}(3)$ lift uniquely to $\mathrm{Spin}(3)$. A Spin structure on manifold also defines a 3-framing up to local change, and a close relationship between linear TQFTs on framed manifolds and projective TQFTs on Spin manifolds ensues.[18]

[17] We will assume here the 4-dualizability of A; proofs will appear in [FST].

[18] This does not apply to *families of manifolds*, because of $\pi_4\mathrm{Spin}(3) = \mathbb{Z}/2$ and the higher groups.

Note that 3-*framings* and *stable framings* are distinct structures, because the inclusion $SO(3) \subset SO$ has index 2 on π_3: there are twice as many local changes of stable framing. Extending a framed theory to stably framed manifolds is always possible, but meets an ambiguity of order 2 (see Theorem 3). Other instances where factors of 2 cause problems, if not tracked correctly, relate Spin(3)-structures, Spin structures and orientations. Here, we bring some order with the explicit descriptions of bordism groups and maps between them.

(B.2) Tangential structures. An *n-dimensional tangential structure* τ is a space equipped with a map to $BO(n)$, and a τ-structure on a manifold is a factorization via τ of the structure map of the tangent bundle. Commonly, this represents a reduction of structure group, such as to the trivial group (an n-framing), the groups $SO(n), \mathrm{Spin}(n) \to O(n)$, but can be defined more generally. Thus, the homotopy fiber of the map representing the Pontrjagin class

$$p_1 : BSO(n) \to \Sigma^4 H\mathbb{Z}$$

defines an oriented p_1-structure, denoted by $SO^{p_1}(n)$. $\mathrm{Spin}^{p_1}(n)$-structures are defined similarly. *Signature structures*, considered in [Tu] for oriented RT theories, are more subtle: see §8.

Bordism groups of n-manifolds with tangential structure are the π_0 groups of the respective *Madsen-Tillmann spectra* $MT(\tau)$, the (unreduced) de-suspensions of the underlying space of τ using the standard representation of the orthogonal group [GMTW]. They are the n-fold looped group completions of the respective bordism categories. The MT spectrum for n-framings is the shifted sphere $\mathbb{S}^{-n}$.

(B.3) Some bordism groups. We describe several 3-manifold bordism groups. The definitional shifts in $MT(n)$ place the 0-manifolds in degree $(-n)$, and account for the shifted indexing of homotopy groups; while in the dimensionally-stable M, points are in degree 0. All copies of $\pi_3^s = \mathbb{Z}/24$ below come from the inclusion of the lowest (dimensionally shifted) sphere $\mathbb{S}$. However, in terms of units of p_1-structure, which normalize all stable summands in the right column, this is better read as $2\mathbb{Z}/48$: a one-unit shift of p_1-structure is not executable on $\mathrm{Spin}^{p_1/2}$-manifolds.

$\pi_1 MT\mathrm{Spin}_2^{p_1/4} = \pi_3^s \oplus \pi_1^s$	$\pi_0 MT\mathrm{Spin}_3^{p_1/4} = \pi_3^s$	N/A
$\pi_1 MT\mathrm{Spin}_2^{p_1/2} = \pi_3^s \oplus \mathbb{Z}/4$	$\pi_0 MT\mathrm{Spin}_3^{p_1/2} = \pi_3^s \oplus \mathbb{Z}/2$	$\pi_3 M\mathrm{Spin}^{p_1/2} = \pi_3^s$
$\pi_1 MT\mathrm{Spin}_2^{p_1} = \mathbb{Z}/48 \oplus \mathbb{Z}/4$	$\pi_0 MT\mathrm{Spin}_3^{p_1} = \mathbb{Z}/48 \oplus \mathbb{Z}/2$	$\pi_3 M\mathrm{Spin}^{p_1} = \mathbb{Z}/48$
$\pi_1 MT\mathrm{Spin}_2^{\mathbb{C}p_1} = \mathbb{C}/48\mathbb{Z} \oplus \mathbb{Z}/4$	$\pi_0 MT\mathrm{Spin}_3^{\mathbb{C}p_1} = \mathbb{C}/48\mathbb{Z} \oplus \mathbb{Z}/2$	$\pi_3 M\mathrm{Spin}^{\mathbb{C}p_1} = \mathbb{C}/48\mathbb{Z}$
$\pi_1 MT\mathrm{Spin}_2 = \mathbb{Z}/4$	$\pi_0 MT\mathrm{Spin}_3 = \mathbb{Z}/2$	$\pi_3 M\mathrm{Spin} = 0$
$\pi_1 MTSO_2^{p_1} = \mathbb{Z}/12$	$\pi_0 MTSO_3^{p_1} = \mathbb{Z}/6$	$\pi_3 MSO^{p_1} = \mathbb{Z}/3$
$\pi_1 MTSO_2^{\mathbb{C}p_1} = \mathbb{C}/12\mathbb{Z}$	$\pi_0 MTSO_3^{\mathbb{C}p_1} = \mathbb{C}/6\mathbb{Z}$	$\pi_3 MSO^{\mathbb{C}p_1} = \mathbb{C}/3\mathbb{Z}$

Table 2: Some relevant bordism groups

A uniform description for the p_1-structured Spin groups is: the sum of the dimensionally-stable group with the same structure, and the p_1-independent group in the same dimension. The orthogonal groups also appear to split in this way, but the map from the Spin groups mixes the stable and unstable summands. We now describe the most important groups more explicitly.

(B.4) Dimension 2*.* A p_1-structure on oriented surfaces trivializes the 12th power of the Hodge determinant bundle $\delta := \det(H^*(\mathscr{O}))$. The 12 (now flat) powers of δ detect the group $\pi_1 MTSO^{p_1}(2)$, and shifts in p_1-structure cycle through them.

A $p_1/4$-structure kills the 4-cell in $B\mathrm{Spin}(2) = \mathbb{CP}^\infty$, leaving, below dimension 4, the desuspension of S^2 by its tangent bundle; the latter is stably trivial, which leads to our split description of $\pi_1 MT\mathrm{Spin}_2^{p_1/4}$. Spin structures define fourth roots of δ from the Pfaffian of the Dirac operator; the four roots differ by flat line bundles, which detect the group $\pi_1 MT\mathrm{Spin}(2)$. Finally, a Spin structure enhances the $\mathbb{Z}/12$ of $\pi_1 MT\mathrm{SO}^{p_1}(2)$ to the $\mathbb{Z}/48$ summand of $\pi_1 MT\mathrm{Spin}^{p_1}(2)$, in addition to $\pi_1 MT\mathrm{Spin}(2) = \mathbb{Z}/4$.

(B.5) Dimension 3. The group $MT\mathrm{Spin}(3)$ is a quotient of π_3^s. The invertible TQFT of order 2 on framed 3-manifolds factors through Spin(3)-structures; we called the result ν in §3.9.

As an invertible theory, ν is uniquely determined by its 3-manifold invariant. We can describe this explicitly. For a closed, Spin 3-manifold N, choose a trivialization of the cocycle $p_1/4$. Also choose a Spin 4-manifold M with $\partial M = N$, exploiting the vanishing of the (dimensionally stabilized) Spin 3-bordism group. The cocycle $p_1/2$ on M has been trivialized on ∂M, so $\int_M p_1/2$ is an integer. A change in the boundary trivialization of $p_1/4$ shifts the integral by an even number, so $\nu(N) := \int_M p_1/2 \mod 2$ is well-defined. It vanishes if we can find a (stable) 3-dimensional reduction of TM, because $p_1/4$ is then an integral class. On the other hand, $\nu \neq 0$ on the 3-sphere with Lie group framing.

(B.6) Framed and stably framed groups. The map $\pi_3\mathrm{SO}(3) \to \pi_3\mathrm{SO}$ is an index 2 inclusion. Because of that, stabilizing the 3-framing on a 3-manifold has the same effect as killing w_2 and $p_1/2$ in SO(3). If we rigidify the bordism 3-category above dimension 3 by collapsing diffeomorphism groups of 3-manifolds, then a $\mathrm{Spin}^{p_1/2}(3)$ structure becomes equivalent to a stable framing.

The same argument shows the equivalence of the framed and $\mathrm{Spin}^{p_1/4}(3)$ strictified 3-categories. On the other hand, the cokernel $\mathbb{Z}/2$ of the map $\pi_3^s \to M\mathrm{Spin}^{p_1}$ is detected by the difference between the mod 2 reduction of the p_1-structure and Sq^2 of the Spin structure, viewed as trivialization of w_2. (The identity $Sq^2 w_2 = p_1 \mod 2$ is respected on structures defined from a framing.)

(B.7) The split Spin *groups.* We have natural isomorphisms induced from the stabilization and p_1-forgetful maps:

$$\begin{aligned} \pi_1 MT\mathrm{Spin}^{p_1}(2) &= \pi_3 M\mathrm{Spin}^{p_1} \oplus \pi_1 MT\mathrm{Spin}(2), \\ \pi_0 MT\mathrm{Spin}^{p_1}(3) &= \pi_3 M\mathrm{Spin}^{p_1} \oplus \pi_0 MT\mathrm{Spin}(3); \end{aligned} \tag{B.8}$$

Because all oriented 3-manifolds admit p_1-structures, the forgetful maps are surjective. Shifts in p_1-structure can be shown to act cyclically on the first summand, so the four, respectively two orbits in the left groups must map bijectively onto the right summands in (B.8), establishing the splitting. Moreover, shifts in p_1-structure cycle through the $\mathbb{Z}/48$ summands only.

B.9 Remark. An alternate description of these groups comes from the following extension sequences (which stem from the divisibility of p_1 by 4 and 2 in the respective Spin groups):

$$\begin{aligned} \pi_3^s \oplus \pi_1^s &\rightarrowtail \pi_1 MT\mathrm{Spin}^{p_1}(2) \twoheadrightarrow \mathbb{Z}/4, \\ \pi_3^s &\rightarrowtail \pi_0 MT\mathrm{Spin}^{p_1}(3) \twoheadrightarrow \mathbb{Z}/4, \\ \pi_3^s &\rightarrowtail \quad \pi_3 M\mathrm{Spin}^{p_1} \quad \twoheadrightarrow \mathbb{Z}/2. \end{aligned}$$

The top one comes from $\mathbb{S}^{-2} \oplus \mathbb{S}^0 = \Sigma^{-TS^2} S^2 \subset MT\mathrm{SO}(2)$, with the source split because TS^2 is stably trivial, the other two come from the inclusion of the lowest degree spheres. The extensions are classified by the order-two elements $(2, 1), 2$ and 1 in the respective Ext groups.

(B.10) Maps between structural groups. The homotopy groups align to 3-manifold structures and fit into a natural diagram of maps induced by relaxing the structures:

$$\begin{array}{ccccccc}
 & \overset{s_2}{\nearrow} & \pi_1 MT\mathrm{Spin}_2^{p_1/2} & \longrightarrow & \pi_1 MT\mathrm{Spin}_2^{p_1} & \xrightarrow{o_2} & \pi_1 MT\mathrm{SO}_2^{p_1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
\pi_3^s \cong \pi_0 MT\mathrm{Spin}_3^{p_1/4} & \xrightarrow{s_3} & \pi_0 MT\mathrm{Spin}_3^{p_1/2} & \longrightarrow & \pi_0 MT\mathrm{Spin}_3^{p_1} & \xrightarrow{o_3} & \pi_0 MT\mathrm{SO}_3^{p_1} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & \underset{\sim}{\overset{s}{\searrow}} & \pi_3 M\mathrm{Spin}^{p_1/2} & \longrightarrow & \pi_3 M\mathrm{Spin}^{p_1} & \xrightarrow{o} & \pi_3 M\mathrm{SO}^{p_1}.
\end{array}$$

B.11 Proposition. *All vertical maps are surjective, the $s_\bullet$ are injective, while all $o_\bullet \circ s_\bullet$ are surjective. More precisely, we can choose generators so that*

$$\begin{array}{ccccccc}
 & \overset{(1,\pm1)}{\nearrow} & \mathbb{Z}/24 \oplus \mathbb{Z}/4 & \xrightarrow{(2,1)} & \mathbb{Z}/48 \oplus \mathbb{Z}/4 & \xrightarrow{(-1,\pm3)} & \mathbb{Z}/12 \\
 & & \downarrow \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right] & & \downarrow \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right] & & \downarrow 1 \\
\mathbb{Z}/24 & \xrightarrow{(1,1)} & \mathbb{Z}/24 \oplus \mathbb{Z}/2 & \xrightarrow{\left[\begin{smallmatrix}2&0\\0&1\end{smallmatrix}\right]} & \mathbb{Z}/48 \oplus \mathbb{Z}/2 & \xrightarrow{(-1,3)} & \mathbb{Z}/6 \\
 & & \downarrow \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] & & \downarrow \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] & & \downarrow 1 \\
 & \overset{=}{\searrow} & \mathbb{Z}/24 & \xrightarrow{2} & \mathbb{Z}/48 & \xrightarrow{1} & \mathbb{Z}/3
\end{array}$$

The sign ambiguity is absorbed by the sign automorphism of $\mathbb{Z}/4$.

Sketch of proof. The groups are determined and compared from the Atiyah-Hirzebruch spectral sequences for *twisted* $I^*_{\mathbb{C}^\times}$-cohomologies of the respective BSpin and BSO groups. The twistings are by the (negative of the) standard representation, acting on coefficients via the J-homomorphism.

We can certainly choose generators of $\mathbb{Z}/24, \mathbb{Z}/48, \mathbb{Z}/12, \mathbb{Z}/6$ and $\mathbb{Z}/3$, compatibly with the maps indicated. Surjectivity of $o_\bullet \circ s_\bullet$ shows that the maps out of the unstable kernels to the rightmost groups must be injective, and the ones out of π_3^s to them must be surjective.

Finally, a p_1-structure on surfaces is trivialization of the 12th power of the Hodge determinant bundle δ, and shifts cycle through the twelve (now flat) powers of δ. The split $\mathbb{Z}/48$ summand must then surject through o_2. Combined with the injectivity of s_3 on $\mathbb{Z}/2$ and the choice of sending the standard generator $1 \in \mathbb{Z}/24$ to the standard ones in the rightmost groups, this pins the maps, up to the sign ambiguity flagged. □

(B.12) The free Fermion. One consequence concerns the free Fermion theory ψ. The standard character of π_3^s can be extended to $\mathrm{Spin}^{p_1}(3)$ in four different ways, as can be seen from Proposition B.11. We choose for ψ the one coming from the stable structure Spin^{p_1}. According to [FH], this is a *reflection-positive* extension. One observation, which can cause much grief in the form of apparent (although not genuine) contradictions, is the following

B.13 Proposition. *When restricted to framed manifolds, the theory $\psi^{\otimes 4}$ factors uniquely through* $\mathrm{SO}^{p_1}(3)$*-structures. When defined on manifolds with* $\mathrm{Spin}^{p_1}(3)$ *structure, only $\psi^{\otimes 16}$ factors. Similarly, $\psi^{\otimes 2}$ factors from framed through* $\mathrm{SO}^{p_1}(2)$*-manifolds, but only $\psi^{\otimes 16}$ descends there from* $\mathrm{Spin}^{p_1}(2)$*-structures.*

This may seem counter-intuitive, as we are factoring through *the same* quotient $\mathbb{Z}/6$: depending on where we start our contemplation, $\psi^{\otimes 4}$ *does/does not* define a theory on $\mathrm{SO}^{p_1}(3)$-manifolds!

Proof. This is clear from the explicit factoring maps in Proposition B.11: all kernels of the right horizontal arrows are isomorphic copies of $\mathbb{Z}/16$. $\square$

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References

[AC] E. Arbarello, M. Cornalba: The Picard groups of the moduli spaces of curves. *Topology* **26**(2) (1987), 153–171

[A] M. Atiyah: Topological quantum field theories, *Publ. Math. IHÉS* **68** (1988), 175–186

[B] A. Beliakova: Spin topological quantum field theories. *Int. J. Mathematics***9** (2) (1998) 129–152

[BDSV] B. Bartlett, C. Douglas, C. Schommer-Pries, J. Vicary: *Modular categories as representations of the 3-dimensional bordism 2-category.* arXiv:1509.06811

[BJS] A. Brochier, D. Jordan, N. Snyder: On dualizability of braided tensor categories. *Compositio Math.* **157** (2021), 435–483

[BHMV] C. Blanchet, N. Habegger, G. Masbaum, P. Vogel: Topological quantum field theories derived from the Kauffman bracket, *Topology* **34**(4) (1995), 883–927.

[BK] B. Bakalov, A. Kirillov: Lectures on Tensor Categories and Modular Functors. *University Lecture Series* **21** (2001)

[BM] C. Blanchet, G. Masbaum: Topological quantum field theories for surfaces with spin structure, *Duke Math. J.* **82**(2), (1996) 229–267

[CF2] T. D. Décoppet, P. Huston, T. Johnson-Freyd, D. Nikshych, D. Penneys, J. Plavnik, D. Reutter, M. Yu: *The classification of fusion 2-categories.* arXiv:2411.05907

[DMNO] A. Davydov, M. Müger, D. Nykshich, V. Ostrik: The Witt group of non-degenerate braided fusion categories. *Journal für die reine und angewandte Mathematik (Crelle's Journal)* **677** (2013), 135–177

[DSPS] C. Douglas, C. Schommer-Pries, N. Snyder: Dualizable tensor categories. *Memoirs of the American Mathematical Society* **268** (1308), (2020)

[DN] A. Davidov, D. Nikshych: Braided Picard groups and graded extensions of braided tensor categories. *Selecta Math.* **27**(4) (2021),

[EGNO] P. Etingof, D. Nikshych, V. Ostrik: Tensor categories. *Mathematical Surveys and Monographs* **205**, AMS (2015)

[ENO] P. Etingof, D. Nikshych, V. Ostrik: Fusion categories and homotopy theory, *Quantum Topology* **1**(3) (2010), 209–273

[FH] D.S. Freed, M.J. Hopkins: Reflection positivity and invertible topological phases. *Geom. Topol.* **25** (2021), 1165-1330

[FT1] D.S. Freed, C. Teleman: Gapped Boundary Theories in Three Dimensions. *Commun. Math. Phys.* **388** (2021), 845–892

[FT2] D.S. Freed, C. Teleman: *Tangential structures and central charges in 3-dimensional Chern–Simons theories.* In preparation.

[FT3] D.S. Freed, C. Teleman: Relative quantum field theory. *Commun. Math. Phys.* **326** (2014), 459–476

[FT. . .] D.S. Freed, C. Teleman, et al: *Fusion super-categories.* In preparation.

[FST] D.S. Freed, C. Scheimbauer, C. Teleman: In preparation

[GMTW] S. Galatius, I. Madsen, U. Tillmann, M. Weiss: The homotopy type of the cobordism category. *Acta Math.* **202**(2) (2009), 195–239

[H] B. Haioun: Non-semisimple WRT at the boundary of Crane-Yetter. `arXiv:2503.20905`

[He] A.G Henriques: What Chern-Simons theory assigns to a point. *Proc. Natl. Acad. Sci. USA*, **114**(51), (2017) 13418–13423

[HPT] A.G. Henriques, D. Penneys, J. Tener: Categorified trace for module tensor categories over braided tensor categories. *Doc. Math.* **21** (2016), 1089–1149

[J] J. Jenquin: *Spin Chern-Simons and Spin TQFTs.* `arXiv:math/0605239`

[JS] T. Johnson-Freyd, C. Scheimbauer: (Op)lax natural transformations, twisted quantum field theories, and "even higher" Morita categories. *Adv. Math.* **307** (2017), 147–223

[K] L. Kong: *How to take a square root of a modular tensor category?* Lecture at BIRS, Oaxaca, September 2018

[KYZZ] L. Kong, W. Yuan, Z.-H. Zhang, H. Zheng: Enriched monoidal categories I: Centers. *Quantum Topology* **16**(2), (2024) 343–417

[LKW] T. Lan, L. Kong, X.-G. Wen: A classification of 3+1D bosonic topological orders: the case when point-like excitations are all bosons. *Phys. Rev. X* **8**, (2018) 021074 1–24

[L] J. Lurie: On the classification of topological field theories. In: *Current Developments in Mathematics* **2009**, International Press, (2009) 129–280

[M] M. Müger: From subfactors to categories and topology. II. The quantum double of tensor categories and subfactors. *J. Pure Appl. Algebra* **180** (2003), 159–219

[NRYZ] S.-H. Ng, E.C. Rowell, Y. Wang, Q. Zhang: Higher central charges and Witt groups. *Adv. Math* **404** (2022), 1–38

[P] D. Penneys: Notes for Math 8800, *Unitary tensor categories*, `https://people.math.osu.edu/penneys.2/8800/Notes/UMTCs.pdf`

[S] G.B. Segal: The definition of conformal field theory. *London Math. Soc. Lecture Note Ser.*, **308**, Cambridge (2004), 421–577

[T] C. Teleman: Gauge theory and mirror symmetry. *Proc. of ICM, Seoul 2014 II*, 1309–1332

[Tu] V. Turaev: *Quantum Invariants of Knots and* 3*-Manifolds*, 3rd ed. *Studies in Mathematics* **18**, De Gruyter GmbH (2016)

[RT] N. Reshetikhin, V. Turaev: Invariants of 3-manifolds via link polynomials and quantum groups, *Invent. Math.* **103**(3) (1991), 547–597

[Wa] K. Walker, *TQFTs*. `https://canyon23.net/math/tc.pdf`.

[W] E. Witten: Quantum Field Theory and the Jones polynomial. *Comm. Math. Phys.* **121**(3) (1989), 351–399