Gauge Theory and Mirror Symmetry

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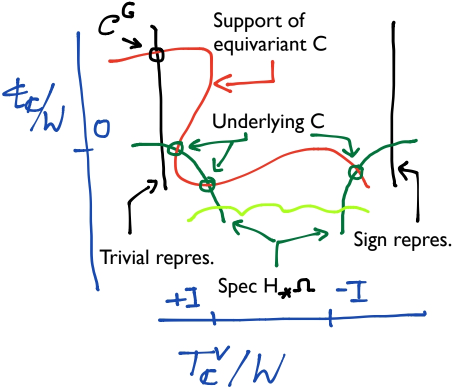
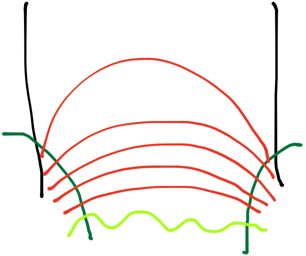
Character space for SO(3) and Toda foliation





$$t_{\mathbb{C}} \ni \zeta = q(z - z^{-1})$$

Gromov-Witten theory

Among topological quantum field theories studied in the past decades, Gromov-Witten theory has enjoyed continued interest.

It associates to a compact symplectic manifold X a *space of states* $H^*(X)$. *Correlators* assigned to surfaces with points labeled by states count the pseudo-holomorphic maps to X with incidence conditions.

For the universal suface, these numbers refine to cohomology classes on the Deligne-Mumford spaces $\overline{M}_g^n$.

These invariants have not yet been classified structurally.

Mirror symmetry (Lerche, Vafa, Warner and refined by many others) promises to reduce GW theory to more standard computations in the complex geometry of a conjectural mirror manifold $X^\vee$.

Homological mirror symmetry (after Kontsevich)

This is a program to spell out the structure of GW invariants. Key idea:

Extend the TQFT to surfaces with corners labeled by *boundary conditions*; those form a linear category with structure ('Calabi-Yau' $=$ cyclic A_∞; plus ...), which should determine all invariants.

Symplectic side, (X, ω): Fukaya's A_∞ category $\mathcal{F}(X)$;
Complex side, $D^b Coh(X^\vee)$ (with its Yoneda structure).

The two structured categories have been matched in many examples: $K3$ (Seidel); del Pezzo surfaces, weighted projective spaces (Auroux, Katzarkov, Orlov); toric Fanos (FO^3+ Abouzaid$+$others); Calabi-Yau hypersufaces (Sheridan)

The mirror of a toric variety X with torus $T_\mathbb{C}$:
the dual torus $T_\mathbb{C}^\vee$, plus a *super-potential* Ψ (a Laurent polynomial).
The associated category of Ψ-*Matrix factorizations* is $\mathbb{Z}/2$-graded.

Group actions and Hamiltonian quotients

Many GW computations involve *Hamiltonian quotients* of simpler varieties. Thus, projective toric varieties are quotients of $\mathbb{C}^n$ by linear torus actions. Their mirrors can be described in those terms.

Example (Givental-Hori-Vafa mirror)

The best-known case is $\mathbb{P}^{n-1} = \mathbb{C}^n /\!/ \mathrm{U}(1)$, with mirror

$$(\mathbb{C}^*)^{n-1} = \{(z_1, \ldots, z_n) \mid z_1 z_2 \cdots z_n = q\}, \Psi = z_1 + \cdots + z_n$$

For $Y = \mathbb{C}^n$, with standard $(\mathbb{C}^*)^n$ action, declare the mirror to be

$$Y^\vee = (\mathbb{C}^*)^n, \quad \Psi = z_1 + \cdots + z_n.$$

For $X = \mathbb{C}^n /\!/ K$ with $K_\mathbb{C} \subset (\mathbb{C}^*)^n$, $X^\vee_\mathbf{q}$ is the fiber over $\mathbf{q} \in K^\vee_\mathbb{C}$ of the dual surjection $(\mathbb{C}^*)^n \twoheadrightarrow K^\vee_\mathbb{C}$, and the super-potential is the restricted Ψ.

($\mathbf{q}$ tracks degrees of holomorphic curves.)

Mirror of a Lie group action I

Mirror symmetry for Hamiltonian quotients raises the following

Basic Questions

1. Find the mirror structure on $X^\vee$ for a Hamiltonian group action on X.
2. Describe the mirror to the GIT quotient.

Basic Answers (Torus case: 0th order approximation)

1. The mirror to a T-action on (X, ω) is a holomorphic map $X^\vee \to T^\vee_{\mathbb{C}}$.
2. (Conj.) The mirror of $X//T$ is the (derived) fiber of $X^\vee$ over 1.

Basic Answers (Compact, connected G; (-1)st order approximation)

1. The mirror to a G-action on X is a holomorphic map from $X^\vee$ to the space of conjugacy classes in the Langlands dual group $G^\vee_{\mathbb{C}}$.
2. (Conj.) The mirror of $X//G$ is closely related to the fiber over 1.

A template: geometric quantization

When I grew up, a symplectic manifold (X, ω) used to quantize to a 1-dimensional field theory (quantum mechanics).

Given a polarized line bundle $\mathcal{L}$ with curvature ω, the space of quantum states is $H^*(X; \mathcal{L})$.

A Hamiltonian G-action on X quantizes to an action on $H^*(X; \mathcal{L})$.
The classical gauged theory is based on the symplectic reduction $X//G$.
The space of states of the gauged quantum theory is $H^*(X; \mathcal{L})^G$.

A famous theorem of Guillemin and Sternberg equates the two ways of gauging, before/after quantization: *quantization commutes with reduction*.

This is very effective in studying symplectic quotients (their topology, K-theory) thanks to our good understanding of the representation theory of a compact group G.

Replicating this success requires a *representation theory of G on categories*.

Topological group actions on categories

Definition (Action of G on a category $\mathcal{C}$)

- An endofunctor Φ_g of $\mathcal{C}$ for each $g \in G$
- A natural isomorphism $\Phi_{gh} \xrightarrow{\sim} \Phi_g \circ \Phi_h$ for each pair,

subject to an obvious coherence condition for all triples (g, h, k).

Remark

1. This is a homomorphism from G to the 2-group of auto-functors of $\mathcal{C}$.
2. *Topological*: add continuity *and* a *trivialization* $\Phi \cong \mathrm{Id}$ near $1 \in G$.
3. A derived version relaxes equalities to coherent homotopies.

Definition (invariant category $\mathcal{C}^G$)

- Objects of $\mathcal{C}^G$: tuples $\{x; \varphi_{x,g} : x \xrightarrow{\sim} \Phi_g x\}_{g \in G} \mid$ coherence condition
- Morphisms of $\mathcal{C}^G$: $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ commuting with the φ.

Topological G-actions up to homotopy

Theorem (Connected G acting topologically on $\mathcal{C}$)

1. *A topological, up to homotopy action of G on $\mathcal{C}$ is the same as An E_2-algebra morphism $C_*(\Omega G) \to HCH^*(\mathcal{C})$. This is also a module structure of $\mathcal{C}$ over $(C_*(\Omega G)$-modules$, \otimes)$.*
2. *The invariant category $\mathcal{C}^G$ is the fiber over $0 \in \mathrm{Spec}\, H_*(\Omega G)$.*

Theorem (X compact)

Hamiltonian G-action on $(X, \omega) \Rightarrow$ topological G-action on $\mathcal{F}(X)$.

Remark

Without E_2, this is essentially due to P. Seidel.

$H_*(\Omega G; \mathbb{C})$ is a fully commutative (E_∞) algebra.
The same is expected of $HH^*(\mathcal{F}(X))$ (at least, when $\cong QH^*(X)$).
But, the E_2 map contains *more data* than the underlying algebra map.

Spectral theory I

We obtain a spectral decomposition of G-categories over $\operatorname{Spec} H_*(\Omega G)$.
That is the dual torus $T_{\mathbb{C}}^{\vee}$ if $G = T$; a vector space for simple G.

Quantization commutes with reduction now predicts $\mathcal{F}(X)^G \equiv \mathcal{F}(X/\!/G)$.

We thus expect $\mathcal{F}(X/\!/G) =$ fiber of $\mathcal{F}(X)$ over $0 \in \operatorname{Spec} H_*(\Omega G)$.
But, examples show that fiber to be *a completion of* $\mathcal{F}(X/\!/G)$, often 0.

This is because the notion of G-action up to homotopy is too weak.

Analogue: Weaken linear G-representations to homotopy (stable) representations; this completes the representation ring $K_G(*)$ to $K(BG)$.

Moral: we must keep G-actions strict; everything else up to coherent G-homotopies.

In particular, the E_2 map $\Omega G \to HCH^*$ must be G-equivariant.

Interpretation: the holonomy representation

The Hochschild cohomologies $\mathbf{R}\mathrm{Hom}(\mathrm{Id};\Phi_g)$ of the Φ_g assemble to a multiplicative complex of sheaves on G.
Local triviality of the G-action converts this to a *local system*.
The ΩG action on HCH^* is its holonomy.
This system is weakly Ad-equivariant. We'll insist on *strict* equivariance.
The $H_*(\Omega G)$-structure on HH^* upgrades to a $H_*^G(\Omega G)$-structure on equivariant HH^*.

Remark

For compact X, $HH^*\mathcal{F}(X)$ is (optimistically) quantum cohomology.
Get over G a multiplicative, Ad-equivariant local system w/ fiber $QH^*(X)$.
The equivariant homology $H_*^G(G;QH^*(X))$ carries a *Pontryagin product*.
QR conjecture $\Rightarrow$ the result is isomorphic to $QH^*(X/\!/G)$.

Inclusion of 1 in G gives a homomorphism $QH_*^G(X) \to H_*^G(G;QH^*(X))$ from *Givental's equivariant quantum cohomology*.
This is Woodward's *quantum Kirwan map*.

The BFM space: Spectral theory II

Here are several descriptions of the space on which G-categories localize:

Theorem (Bezrkukavnikov-Finkelberg-Mirkovic)

1. *Spec* $H_*^G(\Omega G)$ *is an algebraic symplectic manifold, isomorphic to the algebraic symplectic reduction* $T^*_{reg} G^\vee //_{Ad} G^\vee$.
2. *It is an affine resolution of singularities of* $(T^* T^\vee_{\mathbb{C}})/W$.
3. *The fiber of Spec* $H_*^G(\Omega G)$ *over* $0 \subset (\mathfrak{t}_{\mathbb{C}})/W$ *is Spec* $H_*(\Omega G)$, *and is a Lagrangian submanifold.*
4. *Completed there,* $H_*^G(\Omega G) = E_2$ *Hochschild cohomology of* $H_*(\Omega G)$ *(a.k.a. the cotangent bundle.)*

Remark

$E_2 HH^*$ controls formal E_2 automorphisms and deformations of $H_*(\Omega G)$.
An E_2-action of $H_*(\Omega G)$ micro-localizes $\mathcal{C}$ to Spec$(E_2 HH^*)$.
A strict G-action *macro-localizes* this to BFM.

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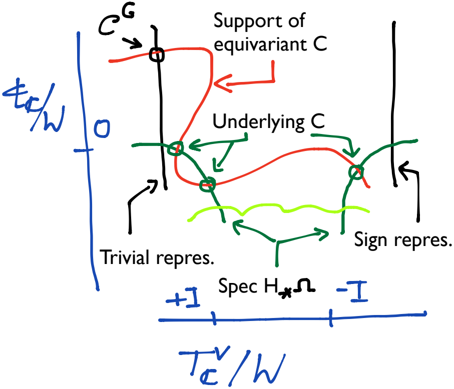

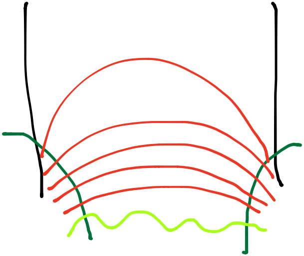

$$\mathfrak{t}_{\mathbb{C}} \ni \xi = q(z - z^{-1})$$

Spectral decomposition via the Toda isomorphism

Schur's lemma fails for categorical representations: irreducible Lagrangians can intersect without agreeing. (No unique irreducible decomposition.)

The following holomorphic Lagrangian foliation of the BFM space appears to serve as spectral decomposition for categorical representations of geometric origin, related to the foliation of a symplectic manifold by co-adjoint orbits under the moment map.

Theorem (Toda isomorphism)

1. $T^*_{reg}G//_{Ad}G \cong N_\chi \backslash\backslash T^*G//_\chi N \quad$ *($\chi =$ regular nilpotent character)*
2. *The cotangent fibers on the right correspond to the Fukaya categories of $G^\vee$-flag varieties.*

Remark

2 contains the computation of mirrors of flag varieties by Peterson, Givental, Kim, Ciocan-Fontanine, and (most comprehensively) Rietsch.