The Haagerup TQFT is not a gauge theory

A recurring conjecture in the math literature, inspired by Moore and Seiberg, states that the Witt equivalence class of every modular tensor category contains a representative Chern-Simons gauge theory of some compact group. While a different source of fusion categories has long been known (the Haagerup construction), a stronger version of the conjecture, asserting braided equivalence instead of Witt equivalence, appears to still circulate. While this stronger conjecture is easily falsified by a numerical check of Frobenius-Perron dimensions, this note gives a human-readable proof that the Haagerup category is a counterexample.

I focus on the original Haagerup fusion category. Six objects, three each of FP dimensions 1 and $(3+\sqrt{13})/2$ give the FP dimension of the tensor category as

$$d = 3 \cdot \left(1 + \frac{11+3\sqrt{13}}{2}\right) = 3\sqrt{13} \cdot \frac{3+\sqrt{13}}{2} \approx 35.725.$$

I claim that d^2, the FP dimension of the Haagerup center, cannot be the dimension of the braided fusion category of Chern-Simons theory of any compact group G: the magnitude and minimal arithmetic at the prime 13 rule out all candidates. The same argumnent with 17 should rule out the Asaeda-Haagerup categories, but I ran out of patience. Those categories have $d = 8\sqrt{17}\cdot(4+\sqrt{17})$.

Specifically, I will check that

- In any gauge theory leading to this dimension, some multiples of 13 must appear among the twistings (levels shifted by dual Coxeter numbers);
- The exceptional groups at the smallest such levels are too large to contribute[1];
- Twisting 39 is too high for any group, and the viable cases at twistings 13 and 26 cannot in any combination acount for the cyclotomic unit (=sin) factorization of d.

The FP dimension D^2 of a category obtained from gauge theory is[2]

$$D^2 = D(G,k)^2 = \pm \frac{\#\pi_0(G)^2 \cdot N}{\Delta\left(\exp \pi \frac{\boldsymbol{\rho}^\vee}{\mathbf{k}}\right)^2}, \tag{1}$$

- N is the number of Verlinde points in the maximal torus of G,
- $\boldsymbol{\rho}^\vee \in \mathfrak{g} \leftrightarrow \boldsymbol{\rho} \in \mathfrak{g}^*$ in the basic inner product (when $\|\alpha_0\|^2 = 2$),
- $\mathbf{k}$ is the vector of twistings,
- $\boldsymbol{\rho}^\vee/\mathbf{k}$ is the vector with suitably normalized components in each simple factor of $\mathfrak{g}$,
- $\pm$ sets the value to positive.

Alternatively, $(\boldsymbol{\rho}^\vee/\mathbf{k}) \leftrightarrow \boldsymbol{\rho}$ under the inner product defined by the quadratic form $\mathbf{k}$, but that formulation conceals the dependence on the level.

The denominator is the volume of the smallest Verlinde conjugacy class. It is the square of

$$\prod_{k;\alpha>0} 2\sin\left(\frac{\pi\langle\alpha|\rho\rangle}{k}\right)$$

factoring over the positive roots, with the respective ρ, k. We have $\langle\alpha|\rho\rangle \leq k-1$, achieved only for the highest root α_0 and at level 0. In the simply laced case, $\langle\alpha|\rho\rangle$ is the number of simple roots in α; for the E series, the sequences with multiplicities are

E_6 : $1^6 2^5 3^5 4^5 5^4 6^3 7^3 8^2 9^1 10^1 11^1$, $\quad E_7$: $1^7 2^6 3^6 4^6 5^6 6^5 7^5 8^4 9^4 10^3 11^3 12^2 13^2 14^1 15^1 16^1 17^1$

E_8 : $1^8 2^7 3^7 4^7 5^7 6^7 7^7 8^6 9^6 10^6 11^6 12^5 13^5 14^4 15^4 16^4 17^4 18^3 19^3 20^2 21^2 22^2 23^2 24^1 25^1 26^1 27^1 28^1 29^1$

[1] Save for E_6 level 1, which is group-like and so of no use.

[2] Except at level zero, when $\pi_1 G$ has torsion.

whence I got $\approx 41,587$ for E_7 and $\approx 119,271$ for E_8. Along with the computation of F_4 and G_2 at the lowest relevant levels (4 and 9), this excludes any helpful appearance of exceptional groups.

Cyclotomic refresher. Note the following factorizations in the cyclotomic ring $\mathbb{Z}[\zeta]$, $\zeta = \exp\frac{\pi i}{13}$:

$$\sqrt{13} = -(\zeta-\bar\zeta)\cdot\dots\cdot(\zeta^6-\bar\zeta^6) = \prod\nolimits_{j=1}^{6} 2\sin\frac{j\pi}{13}$$

$$(*)\qquad \frac{13+3\sqrt{13}}{2} = (\zeta^2-\bar\zeta^2)^2(\zeta^5-\bar\zeta^5)^2(\zeta^6-\bar\zeta^6)^2 = \left(2\sin\frac{2\pi}{13}\cdot 2\sin\frac{5\pi}{13}\cdot 2\sin\frac{6\pi}{13}\right)^2$$

$$\frac{13-3\sqrt{13}}{2} = (\zeta-\bar\zeta)^2(\zeta^3-\bar\zeta^3)^2(\zeta^9-\bar\zeta^9)^2 = \left(2\sin\frac{\pi}{13}\cdot 2\sin\frac{3\pi}{13}\cdot 2\sin\frac{4\pi}{13}\right)^2$$

The sine expressions are *unique*: the factors $2\sin(\pi r/13)$, $1\le r\le 6$, are linearly independent in the group $\mathbb{Q}[\zeta]^\times$. With 52nd or 104th roots included, relations between the respective sines are generated from the doubling formula $\sin(2x) = 2\sin x\cdot\sin(\frac{\pi}{2}-x)$ and the symmetries of sin.

Incidentally, relevant to the Asaeda-Haagerup factor we have

$$17-4\sqrt{17} = \left(2\sin\frac{\pi}{17}\cdot 2\sin\frac{2\pi}{17}\cdot 2\sin\frac{4\pi}{17}\cdot 2\sin\frac{8\pi}{17}\right)^2,$$

with 3, 5, 6 and 7 giving the conjugate $17+4\sqrt{17}$.

Proof of mismatch. If $d = D$, then $\pi_0 G = 1$ or 3, on divisibility grounds in (1). We see from

$$(2)\qquad 3\sqrt{13}\cdot\frac{\sqrt{13}+3}{2}\cdot\prod 2\sin\left(\frac{\pi\langle\alpha|\rho\rangle}{k}\right) = \#\pi_0 G\cdot\sqrt{N}$$

that $13|N$. Squaring leaves overt roots of 13 on the left, but none on the right. The ks must then include multiples of 13, in more than one sin factor. So $N = 13^2 M$, and from (*) we get

$$(3)\qquad 3\cdot\prod_{\alpha>0} 2\sin\left(\frac{\pi\langle\alpha|\rho\rangle}{k}\right) = \#\pi_0 G\cdot\sqrt{M}\cdot\left(2\sin\frac{\pi}{13}\cdot 2\sin\frac{3\pi}{13}\cdot 2\sin\frac{4\pi}{13}\right)^2.$$

It turns out that we cannot replicate the right-hand combination of sines from any group; but for ease, I'll use the magnitude of D to reduce to just a few checks.[3] The table below shows that $\sin\frac{4\pi}{13}$ cannot appear at an acceptably low level and rank. We need $\ell\ge 5$ (and also $n\ge 6$) for SU(n); $\ell\ge 6$ and $n\ge 9$, or else $\ell\ge 8$ and $n\ge 7$, for Spin(n); and $\ell\ge 4$ for Sp(n). The estimate $|\sin x|\le|x|$ suffices to rule these out, along with all higher levels (D increases with the level).

Classical groups at low levels ℓ; $n = k-\ell$.

Group	$\ell = 1$	2	3	4	5
SU(n)	$k-1$	$\frac{\sqrt{k(k-2)}}{2\sin\frac{\pi}{k}}$	$\frac{k\sqrt{k-3}}{(2\sin\frac{\pi}{k})^2\cdot 2\sin\frac{2\pi}{k}}$	$\frac{k^{3/2}\sqrt{k-4}}{\prod_{j=1}^{3}(2\sin\frac{j\pi}{k})^{4-j}}$	$\frac{k^2\sqrt{k-5}}{\prod_{j=1}^{4}(2\sin\frac{j\pi}{k})^{5-j}}$
Spin($n+2$)	2	$2\sqrt{k}$	$\sqrt{k}/\sin\frac{\pi}{2k}$	$\frac{2k}{(2\sin\frac{\pi}{k})^2}$	$\frac{2k}{\prod_{j=1}^{4} 2\sin\frac{j\pi}{2k}}$
Sp($n-1$)	$\frac{\sqrt{2k}}{2\sin\frac{\pi}{k}}$	$\frac{2k}{\prod_{j=1}^{4} 2\sin\frac{j\pi}{2k}}$	$\frac{k^{3/2}}{\prod_{j=1}^{6}\left(2\sin\frac{j\pi}{2k}\right)^{\varepsilon(j)}}$ $\varepsilon(j) = 2,2,1,2,1,1$	$\frac{(2k)^2}{\prod_{j=1}^{8}\left(2\sin\frac{j\pi}{2k}\right)^{\varepsilon(j)}}$ $\varepsilon(j) = 3,3,2,2,2,2,1,1$	way too big

[3]These can be pared further, by incorporating divisibility by 3 into the discussion.

Argument from magnitude. If arithmetic makes you unhappy, one can just argue on the size of D with a few additional checks. One must allow possibile reductions by dividing out a finite central subgroup Z in a product of factors; D drops by a factor of $\#Z$.[4]

For a fixed group, numbers grow in the level ($D \sim C \cdot k^{\dim \mathrm{G}/2}$; N alone ensures a lower bound in scaling, as in $k^{\mathrm{rank}/2}$). The table also shows growth in the rank, for $\ell = 1, \ldots, 5$. The following numbers **rule out semi-simple ranks above** 2 when $13|k$, save for very low levels:

- $\mathrm{SU}(4) = \mathrm{Spin}(6)$ level 5, $D \approx 58.93$;
  level 7 will be out of reach, even after reduction;
- Spin(7), level 5, $D \approx 96.9$;
- Sp(3) level 4, $D \approx 105$.

In low rank,

(1) $\mathrm{SO}(3),\ k = 26$: $D = \sqrt{13}/2 \sin \frac{\pi}{26} \approx 14.9562$, no reduction possible.
(2) $\mathrm{SU}(2),\ k = 13$: $D = \sqrt{13/2}/\sin \frac{\pi}{13} \approx 10.65$, reducible to $\sqrt{13}/2 \sin \frac{\pi}{13} \approx 7.533$.
(3) $\mathrm{SU}(2),\ k = 39$: $D = \sqrt{39/2}/\sin \frac{\pi}{39} \approx 54.87$, reducible to 38.8.
(4) $\mathrm{SO}(3),\ k = 52$: $D \approx 42.225$ too large, as are all higher levels.
(5) $\mathrm{SO}(4)$, $k = (13, 13)$: $D = 13/4 \sin^2 \frac{\pi}{13} \approx 56.74$. Spin(4) is double.
(6) $\mathrm{SU}(3),\ k = 13$: $D \approx 105.749$. Can be reduced by a factor of $\sqrt{3}$ but no help.
(7) $\mathrm{Sp}(2),\ k = 13$: $D = 26/\prod_{j=1}^{4} 2 \sin \frac{\pi j}{26} \approx 341.84$;
(8) $G_2,\ k = 13$: $D \approx 477$;

Viable are cases (1) and (2), with $G = \mathrm{SO}(3) \times H$ and $D_H \approx 35.725/14.95 \approx 2.4$, and respectively in a configuration $G = \mathrm{SU}(2) \times_{\{\pm 1\}} H$, with dimension $D = \frac{1}{2} \cdot 10.65 \times D_H$, so $D_H \approx 6.72$. But no factors that small exist that involve $\sin \frac{\pi}{13}$.

The viable low-level options, at $k = 13$ and 26, are

(9) SU(11), $\ell = 2$: $D = \sqrt{143}/2 \sin\left(\frac{\pi}{13}\right) \approx 24.984$,
and one can factor out $\sqrt{11}$ such as for U(11) to get $\sqrt{13}/2 \sin \frac{\pi}{13} \approx 7.533$;
(10) Sp(11), $\ell = 1$: $D = \sqrt{13/2}/\sin \frac{\pi}{13} \approx 10.653$; one can factor out a $\sqrt{2}$ to get 7.533;
(11) Spin(12), $\ell = 3$: $D = \sqrt{13}/\sin \frac{\pi}{26} \approx 29.91$; once can take out a factor of 2;
(12) Sp(24) level 1: $D = \sqrt{13}/\sin \frac{\pi}{26} \approx 29.91$, and one may factor out 2;

they are excluded by the same lack of tiny co-factors.

Some non-viable or useless options for your enjoyment:

- Sp(10) level 2: $D \approx 341.84$
- SU(10) level 3: $D \approx 191$; one may factor out a $\sqrt{10}$ to get 62.05....
- SU(12) level 1: $D = \sqrt{12}$ (no use)
- Spin(13) level 2: $D = 2\sqrt{13}$ (no use)
- Spin(14) level 1: $D = 2$ (no use)
- Spin(25) level 3: $D = \sqrt{26}/\sin \frac{\pi}{52} \approx 84.45$;
- Sp(37) level 1: $D = \sqrt{39/2}/\sin \frac{\pi}{39} \approx 54.87$, reduction by $\sqrt{2}$ possible to 38.8;
- Spin(38) level 3: $D = \sqrt{39}/\sin \frac{\pi}{78} \approx 155$

[4] The twisting may preclude reduction before adding cofactors, and the drop may be limited by $\sqrt{\#Z}$, as in $\mathrm{SU}(n) \to \mathrm{U}(n)$.