

Coulomb branches for disconnected groups

[Or, how to lose your audience with the first slide]

Five Topics converge here

- (i) Role of Coulomb branches in gauging 2D non-semisimple TQFTs
- (ii) Condensation! From algebras to modules
- (iii) π -crossed braided $\otimes$ categories
- (iv) Higher String Topology
- (v) Matrix Factorizations. [I lied in (iii)]

Ingredients:

compact Lie group G

connected

$$1 \rightarrow G_1 \rightarrow G \rightarrow \pi \rightarrow 1$$

finite

For demonstration

$$1 \rightarrow N \rightarrow H \rightarrow \pi \rightarrow 1$$

Key notion for (i)

Topological group actions on ~~(higher)~~ categories

= actions trivialized on a contractible neighborhood of 1.

Decategorified model: Derived local systems over BG

Can use local systems of DG categories as an approximation

(ii) Condensation

Inside a TQFT, can have

"embedded defects"

"extended operators"

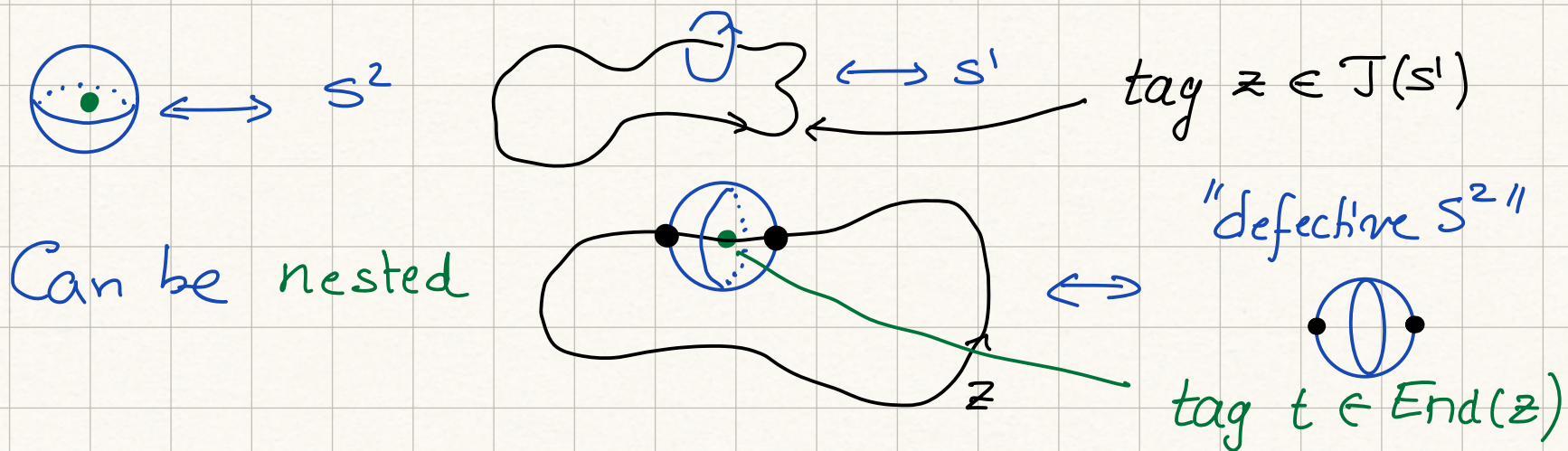
"higher form symmetries"

Example: 3D gauge theory has defect lines

- Wilson defects $\text{Rep}(G)$ labeled
- 't Hooft defects $\text{Vec}(G/\text{Ad}G)$ labeled
- mixed defects $\text{Vec}^G(G)$ labeled

Defects are labelled by quantization of their linking spheres

[Operators of codim. $\ell \leftrightarrow \text{objects} \in \text{Hom}(\mathbb{1}; \mathcal{T}(S^{\ell-1}))$]



Some higher-dimensional defects arise by condensation of lower-dimensional ones



A defect is condensed if its support can be dimensionally reduced

(iii) π -crossed braided fusion categories

In nondegenerate case:

datum needed to gauge the RT theory based on $\mathcal{B}_1$ by π

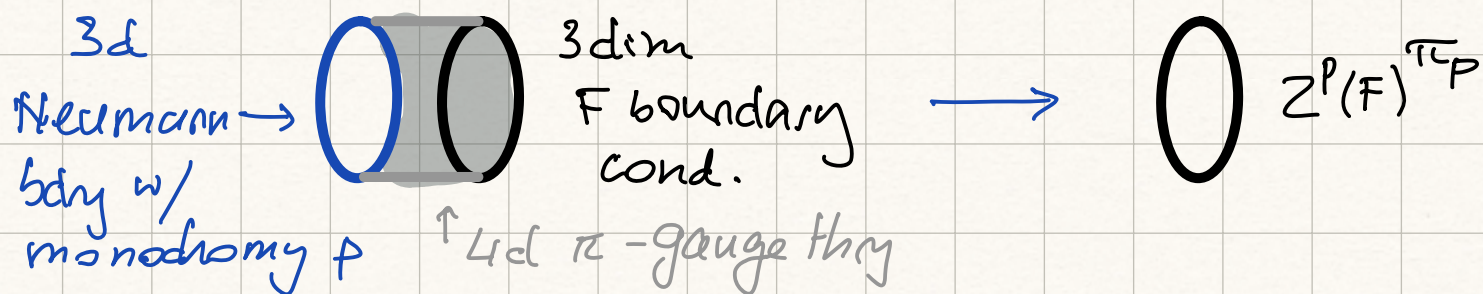
Example $\mathcal{B}_1 = \mathcal{Z}(F)$ F fusion category

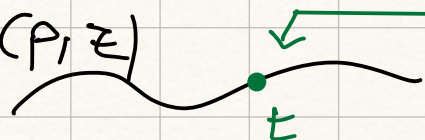
A π -action on F ($\pi \rightarrow \text{BrPic}(F)$) defines $\pi \ltimes F$

$$\mathcal{Z}(\pi \ltimes F) = \left[\bigoplus_{p \in \pi} \mathcal{Z}^p(F) \right]^\pi = \left[\bigoplus_{p \in \pi} \text{Hom}_{F-F}(F, pF) \right]^\pi$$

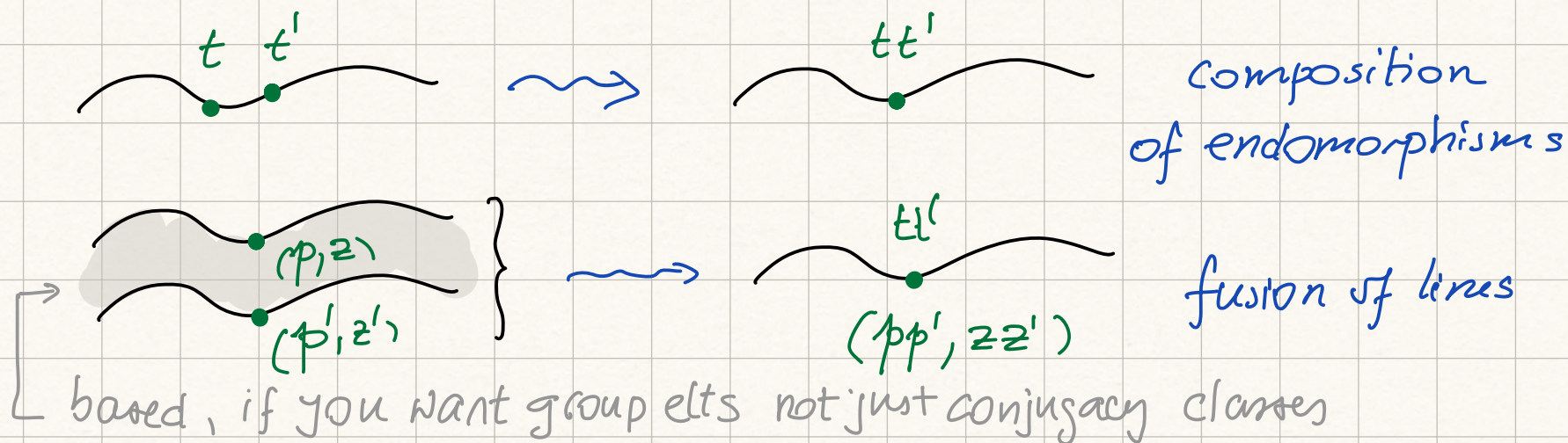
$\uparrow$ twisted sectors $\uparrow$ π -crossed braided

Remark $\mathcal{Z}^p(F)$ is invertible as $\mathcal{Z}$ - $\mathcal{Z}$ bimodule
(Davydov, Etingof, Nykshich, Ostrik)



(p, z)  embedded point operator, tag $t \in \text{End}(z)$
line operator (p, z) $p \in \pi / \text{Ad}$, $z \in \mathcal{Z}^p(F)^{\pi_p}$

Operations $\longleftrightarrow$ π -crossed braided structure



Example

$$1 \rightarrow N \rightarrow H \rightarrow \pi \rightarrow 1$$

$F := \text{Vec}_N$ (N -gauge theory)

H defines an action of π on F

$$\pi \ltimes \text{Vec}_N \cong \text{Vec}_H$$

$$Z(\text{Vec}_H) \cong \int^{\pi} \bigoplus_{p \in \pi} \text{Vec}_p^N(N)$$

" $\text{Vec}^H(H)$ $\uparrow$ p -twisted conjugation

Key Disappointment in Fusion case

Point operators (embedded or not) are boring:

scalars, if the line operator is simple.

[algebraically closed case]

Nothing interesting arises by condensing points

Key feature in Topological gauge theory - connected case

Point operators determine everything

[Case of O matter: general case not completely understood]

Disconnected groups $1 \rightarrow G_1 \rightarrow G \rightarrow \pi \rightarrow 1$

Combine point ops from G_1 and line ops from π .

(iv) Higher String Topology

(~ 2000) Chas & Sullivan: 2D "topless" TQFT from "sigma-model" in a compact oriented manifold M

generated by $C_* \Omega M$ E_1 dg algebra
 $C^* M$ $E_\infty \rightsquigarrow E_1$ dg algebra when $\pi_1 M = 0$
 Koszul duality identifies $HH^* C_* \Omega M \cong H_* LM \cong HH^* C^* M$
 E_2 dg algebra, space of states

Category of "branes": $N \rightarrow M$ (compact manifold)
 (Blumberg, Cohen, T) $\text{Hom}(N, N') := C_* \mathcal{P}_N^{N'} M = C_* \left(N \times_M^h N' \right)$
 [Can form a 2-category of such things for varying M]

Note: Alternative attempt with $C^* \mathcal{P}_N^{N'} M = C^* N \overset{L}{\otimes}_{C^* M} C^* N'$
 runs into problems w/ Eilenberg-Moore convergence
 unless one restricts to simply connected M ...

3D & higher d versions: $C_* \Omega^{d-1} M$ E_{d-1} algebra
 $C^* M$ $E_\infty \rightarrow E_{d-1}$ alg

only equivalent if M $(d-1)$ -connected

Expected recipe Quantize $\pi_1, \dots, \pi_{d-2}$ to a $(d-1)$ category
 enrich with $C_* \Omega^{d-1} M$ at the top

Dimension 3 (2-sphere topology) $M = BG$

Gauge $C_* \Omega^2 M$ by π_1 action: $\pi_1 \ltimes (C_* \Omega^2 M - \text{Mods})$
tensor dg category
on 2-cat of linear dg Cats
w/ central $\Omega^2 M$ action

Point operators from $C_* \text{Map}(S^2; M)$, E_3 dg algebra
 $= \text{End}(\text{Id})$ in $Z(C_* \Omega^2 M - \text{Mods})$

Line operators Condensed, from $C_* L^0 M$ (no π_1)
New from $\mathfrak{p} \in \pi_1 M$

Embedded point operators  $p \in \pi_1 M$
tag $t \in C_* \Omega \Omega_p M - \text{modules}$

Observe:

- (1) Get a (non-semisimple) π_1 -crossed braided $\otimes$ Cat
- (2) If $M = BG$, could expect to get a model for 3D gauge theory

Good news: $M = BG \Rightarrow \pi = \pi_1$ finite (G compact)

Bad news: Expectation (2) is wrong.

(i) Coulomb branches [3D, $N=4$ SUSY G-gauge theory]

[component of the moduli space of vacua]

- * Conjectured (physics) to be a Hyperkähler mfold $\sim T^*T_C^V/W$
- * Constructed by Toda (decades ago) in integrable systems
- * Bezrukavnikov, Finkelberg, Mirkovic (2003): topological def
- * HK structure - Bielawski (rediscovered by Witten, Nakajima, ...)
- * Connection to pure gauge theory developed (T, 2011-)

$$* \mathcal{C}(T) = T^* T_C^V \xrightarrow{\text{proj.}} \mathbb{A}_1^* T_C^V \quad \begin{array}{l} \text{Holomorphic} \\ \text{Integrable System} \end{array}$$

$$* \mathcal{C}(G) = \text{affine resolution of singularities of } T^*T_C^V/W \\ \cong H_*^G(\Omega G) \text{ (B.F.M.)}$$

$$\text{Projects to } \mathcal{G}_C / \underset{\text{Ad}}{G_C} \cong \mathbb{A}_1^* / W = \text{Spec } H^*(BG)$$

The Meta Theorem (topological limit of)

- * 3D $N=4$, topologically twisted SUSY gauge theory for G
 $\iff$ Rozansky - Witten theory of the Toda space for G^V .

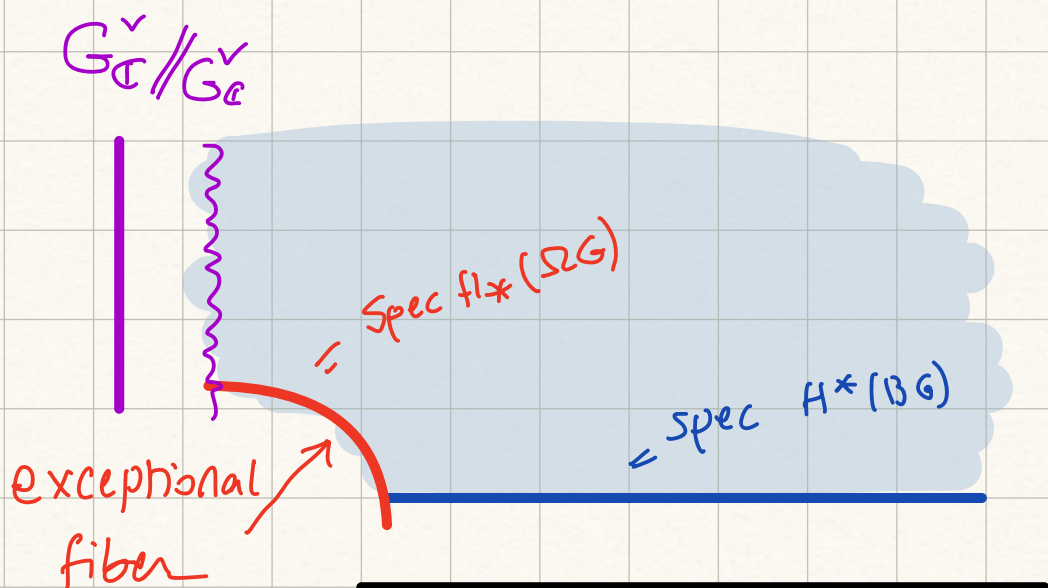
- * Boundary conditions $\sim$ topological categorical representations
Character calculus: holomorphic Lagrangian geometry
of the Toda space $\mathcal{C}(G)$

[Underlying TQFTs and Gauged TQFTs are computed
as intersections with a Dirichlet Neumann Lagrangian.]

- * Line operators are (derived) Coherent sheaves.

Wilson: on D 't Hooft: on N

Example: $SU(2)$



= support of Wilson ops
= Dirichlet boundary

$$\sigma_c / G_c = t_c / w$$

= Neumann boundary

[Fiber over $0 \in t_c$ = Dirichlet Unit section = Neumann]

- * Point ops: functions
- * Line operators labeled by coherent sheaves
- * Boundary conditions: Objects in Kapustin - Rozansky Saulina 2cat

unit section

= support of 't Hooft ops

TQFT Summary

(3) Pure 3D gauge theory "has" an (incomplete), SUSY protected topological sector, T_G .

2D A-models w/ G -symmetry are boundary theories

(2) The ring of point operators is $T_G(S^2) = \mathbb{C}[\ell(G)]$
(E_3 algebra)

(1) $D\text{Coh}[\ell(G)]$ has a braided $\otimes$ structure [Rozansky; Kapranov]
(should be $T_G(S^1)$)

(0) Generator $T_G(\text{pt})$ should be the 2-category for $\ell(G)$
described by Kapustin - Rozanski - Saulina

[Only special cases known, eg integrable systems]

see also [T], Miami 1/25 for proposal

Disconnected groups, $G_1 \rightarrow G \rightarrow \pi$

Obvious model for $\mathcal{T}_G(S')$:

assemble the twisted sectors $Z_{G_1}^p(S')$, $p \in \pi$, into a π -crossed braided tensor cat.

Changes from finite case

(1) For finite groups, $\mathcal{T}_G(S^2) = \mathbb{C} \Rightarrow$ CANNOT reconstruct the theory from its point operators

Connectivity of G_1 allows that:

$$\begin{array}{ccc} \mathcal{T}_G(S^2) - \text{Mod} & =: & \mathcal{T}_G(S') \\ E_3 \text{ algebra} & & E_2 \text{ category} \end{array}$$

Instance of Condensation: point operators $\rightarrow$ line operators

(2) With matter included, the BTC $\mathcal{T}_{G_1, E}(S')$ is proposed to be $\text{D}^b\text{Coh}(\text{desingularization of ?}) \mathcal{C}(G_1, E)$

(3) The twisted sectors $\mathbb{C}[\mathcal{C}^p(G_1, E)] = \mathcal{T}_{G_1}[\bigoplus_p]$ are E_1 but induce π -crossed braided structure on their module cats.

$$\mathcal{T}_{G_1}[\bigoplus_p] \otimes \mathcal{T}_{G_1}[\bigoplus_q] \rightarrow \mathcal{T}_{G_1}[\bigoplus_{p \otimes q} p \otimes q]$$

(4) I lied (to be continued)

What is $\mathcal{I}_{G_1}[\bigoplus_p] ? \subset [\mathcal{L}^p(G_1; E)] ?$
 $H_*^{G_1}(\Omega G_1)$

Interpret $G_1 \backslash \Omega G_1 / \text{Ad} \sim \text{Bun}_{G_1, \mathbb{C}}(\mathbb{P}^1)$

for $\mathcal{L}^p(G_1)$, use bundles w/ monodromy $p \in \pi$ at $0, \infty$

If $\Lambda \subset G_1$ is a Levi subgroup fixed by (a lift of) p

This gives $H_*\left(\Lambda^p \backslash \mathbb{P}^1 \mathbb{C}^p G_1 / \Lambda^p\right)$ with Hecke multiplication (E_1)

Module category is independent of Λ .

Remark: For $\mathcal{L}(G; E)$ need the holomorphic version

Iwahori $\backslash G_{1, \mathbb{C}}((z))^{\leftarrow p\text{-twisted loops}} / \text{Iwahori}$

$$\mathcal{L}(G; V \oplus V^*) = H_*^G(\Omega G; \text{coefficients defined from } V)$$

Coefficients downshift (desuspend) the stratum of $\gamma \in \Omega G$

$$V[[z]] \xrightarrow{\gamma - \text{Id}} V((z)) / V[[z]]$$

[Essentially: Atiyah index sheaf of V -bundle on $\mathbb{P}^1 \times \Omega G$]

(v) Matrix Factorizations

Where I lied: the bimodule categories above
are NOT invertible in general

Example: $G = O(2)$

Untwisted sector: $\mathbb{C}[\bar{z}, \tau]\text{-Mod}$ ($\mathcal{C}(U(1)) = \mathfrak{gl}_1^V \ltimes \mathfrak{gl}_1$)
(Braided structure from $\frac{dz}{z} \wedge d\tau$ form)

But twisted sector is $\mathbb{C} \oplus \mathbb{C}\text{-Mod}$!

Squau is localization of $\mathbb{C}[\bar{z}, \tau]\text{-Mod}$ at $(+1, 0)$.

Remedy: Fix the mistake!

That was not the correct twisted sector description

Alternative description of the twisted sector category:

$\text{Loc}^{G_1}(G_1)$ untwisted, $\text{Loc}^{G_1}(pG_1)$ twisted
= Fixed-point categories for topological G_1 -actions

They admit alternative MF descriptions as moduli cats
over $\mathcal{C}(G_1)$: for $U(1)$, $\text{MF}(2\pi \cdot \log(\pm z))$

over the unramified covering of GL_1^V to its universal cover

Now over a complex manifold X , with map $W: X \rightarrow \mathbb{C}$,
 $MF(X; W)$ is NOT invertible as an $\mathcal{O}_X$ -module category
 ($X = \mathcal{C}(G)$ here)

But it is invertible as an object in $KRS(T^*X)$ with
 the fiberwise group law!

$$MF(X; W) = \text{Hom}_{KRS(T^*X)} \left(\underset{\substack{\uparrow \text{unit object}}}{\mathcal{O}_X}; \underset{\substack{\downarrow \text{supported on } \Gamma(dW)}}{\mathcal{O}_{\Gamma(dW)}} \right)$$

Failure of invertibility comes from taking Hom with unit:

$\mathcal{A}$ invertible $\not\Rightarrow \text{Hom}(\mathbb{1}, \mathcal{A})$ invertible over $\text{End}(\mathbb{1})$.

- Open:
- Spell out " π -crossed braided objects internal to a higher category" as TQFT?
 - Construction from $\mathcal{I}_G(\ast)$?
 Obvious candidate: version of $\pi \times (\text{Loc}(G_1), \ast)$
 Proskematic when adding 'matter' to $\mathcal{C}(G)$:
 (Exit path category for a constructible system / G_1)
 - More basic when adding matter:
 not clear $D^b\text{Coh}[\mathcal{C}(G; E)]$ is the correct $\mathcal{I}_G(S')$
 (Desingularizations?)

Gauge theory with linear matter: E Quaternionic

a hyperkähler space $\mathcal{C}(G; E)$ birational to $\mathcal{T}(G^v)$

First constructed by Braverman - Finkelberg - Nakajima
(as a complex Poisson space) for $E = V \oplus V^*$
 $H_*^G(\Omega G; \text{coefficients built from } V)$.

Hyperkähler structure proposed by Bielawski - Foscolo

Description from $\mathcal{C}(G)$ (& GLSM for V (T, 2018):)
minimal enlargement of $\hat{\mathcal{T}}$ regularizing the Lagrangian section

$$\exp[dW(\tau)]: \quad W(\tau) = \text{Tr}_V \left(\tau (\log \tau - 1) \right) \\ \text{[GLSM superpotential]}$$

"The modification of $\mathcal{C}(G)$ which makes V into a
finite boundary theory (finite, $\text{rk } 1$ over the base)"

That section was also predicted [T] to be the
equivariant symplectic cohomology of V :

Proved (Gonzalez - Mack - Pomerleano), reformulated as

" $\mathbb{C}[\mathcal{C}(G; E)]$ is the subring of $\mathbb{C}[\mathcal{C}(G)]$ which preserves
the $QH_G^*(V)$ lattice in $SH_G^*(V)$ "

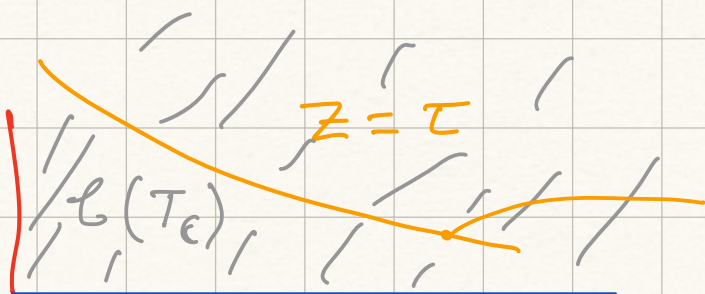
Example

with matter:

$$U(1) \text{ with } E = \mathbb{C} \oplus \overline{\mathbb{C}}$$

$$D = T_{\mathbb{C}}^{\vee}$$

z



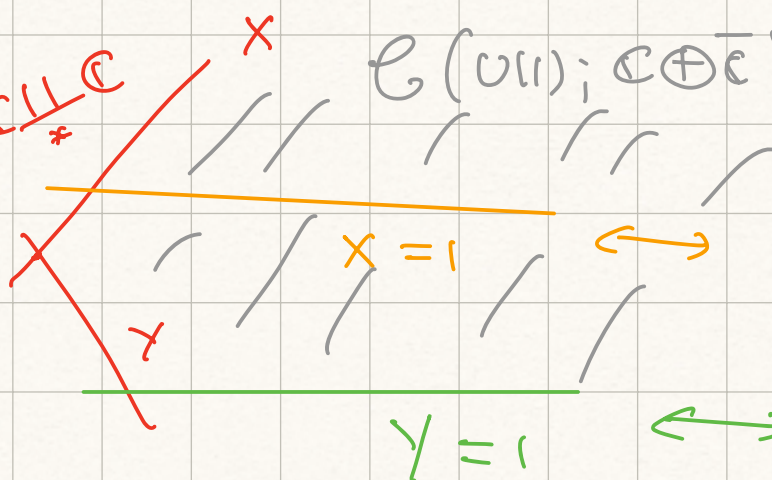
$T \in t_{\mathbb{C}}$ (Neumann)

$$SH_{U(1)}^*(\mathbb{C}) = \mathbb{C}[z, z^{\vee}] = \mathbb{C}[\tau, \tau^{\vee}]$$

representation $\mathbb{C}$

as boundary theory for $T_{U(1)}$

$$D = \mathbb{C} \oplus \mathbb{C}^*$$



$$l(U(1); \mathbb{C} \oplus \overline{\mathbb{C}}) = \text{Spec}[\mathbb{C}[x, y, \tau] / (xy = \tau)]$$

$$QH_{U(1)}^*(\mathbb{C}) \cong \mathbb{C}[\tau]$$

$$QH_{U(1)}^*(\overline{\mathbb{C}}) \cong \mathbb{C}[\tau^{\vee}]$$

$T \in t_{\mathbb{C}}$, Toda base

For $\mathcal{C}(G; E)$ (with matter fields in E)

(2) $\mathbb{C}[\mathcal{C}(G; E)]$ is the chiral ring (point ops)

(1) $T_{\mathbb{C}}(S') = D^b \text{Coh}[\mathcal{C}(G; E)]$ or a desingularization?

(0) not really known (proposals Hilburn-Gammag)