Lecture Topics

- Cobordism Hypothesis – see David

Key Example: Finite Htpy types $\longrightarrow$ n-Alg [variants]?

Correspondence cat & Quantization functor

Induction recipe, up to dim 4 mb.?

Boundary Conds. Dirichlet cond.

EM duality in Abelian case, examples of groups

[Self-dual theories in dim $(4k-1)$]

- EM duality. Freudenthal spectra. – stable htpy range Examples

Htpy type $\longrightarrow$ QFTs not "full" (more symmetries)

Flipping half the span down to a minimal version

Dirichlet cond: gives back a htpy type

Symmetries. Symmetries of QFTs.

Extended ops from defects. Defect calculus.

FH symmetries from group homs.

Normal bdle role in defect calculus.

Anomalous symmetries & examples (Zesting)

- Topological Condensation.

1 fold dep. Ripping open theory

Higher cond (conjectural)

Embedded Dirichlet cond: need for defects

Cond to top dimension - examples

- Lattice models from TQFTs & defects
  - Ising theory. Landau paradigm. KW duality in ∂ of EO1
  - Toric Code. Other TV models. RT models

I TQFTs : fully local $\Rightarrow$ valued in a linear n-category with duals

$$+ \longmapsto Z(+)$$
$$- \longmapsto Z(-) \cong Z(+)^{\vee}$$

duality units & co-units have duals all the way to top dim.

$$X \otimes X^* \xrightarrow{ev} 1 , \quad 1 \xrightarrow{u} X^* \otimes X , \quad \text{relation}$$

In invertible case : dual = inverse

Caution at top : if asking for duals $\Rightarrow$ inverses

$\Rightarrow$ all objects below were invertible

[Useful in proving invertibility].

Main example (in higher dim) – FHT

X finite htpy type $\underset{\text{path integral}}{\overset{\text{finite}}{\rightsquigarrow}}$ $\mathcal{Q}_{m-1}(X,\tau)$ an $(m-1)$ category

$m \geq 0$ generates an m dim TQFT

$\tau \in h^m(X; \mathbb{C}^*)$

Plan: explain relation betw. htpy types and $\mathcal{Q}_m$

[Key ingredient : EM duality]

Illustrate symmetry & defect calculus

Condensation of defects

Applications to lattice models & questions.

Key open question : good target for TQFTs & $\mathcal{Q}_m$

Conflicting desirable properties!

Some wishes:

- linearity
- algebra objects, bimodules should remain cat level
- finite co-invariants / invariants should exist and be iso

  (finite colimits? "Idempotent completion")

- later: units. Generated by its topological ∂ condition

Eg $G \curvearrowright \mathrm{Vect}$ (trivial)

coinv. cat: crossed product

same objects, add iso $x \xrightarrow[\sim]{\varphi_g} gx$ + rels.

Result: free $\mathbb{C}[G]$-modules

Invar: $(x, \varphi_g : x \to gx$ + coherence$)$:

Result: $\mathrm{Rep}(G)$

Have map $\mathrm{Vect}_G \xrightarrow{\mathrm{Norm}} \mathrm{Vect}^G$

Need Idempotent compl to get iso.

1. Θ_m by induction:

(Kontsevich)

$$m=0: \quad \Theta_{-1}(X) = \sum_{p \in \pi_0 X} \prod_i \# \pi_i(X,p)^{(-1)^i} \cdot \tau(p)$$

[Only possible functorial invariant]

X: built from homotopy groups "stacked up"

$\pi_0, \pi_1, \pi_2 \ldots$

Recipe: iterated crossed product.

$$\Theta_m(X) = \bigoplus_{p \in \pi_0 X} \Theta_m(X_p)$$

$$Q_0(X) = \bigoplus \mathbb{C}_p$$

Q_m (connected X) : $Q_{m-1}(\Omega X)$

hey it's a group $\Rightarrow$ algebra object

$\Rightarrow$ linear in higher cat

Alg. Alg. Alg . .

Eg $m = 0$ $\quad Q_0(X) = \bigoplus_{p \in \pi_0 X} \mathbb{C}_p = \mathbb{C}[\pi_0 X]$

$\mathbb{C}\pi_0 X$

$Q_1(X)$: $Q_0(\Omega X) = \bigoplus_{p \in \pi_1} \mathbb{C}_p$

hey it's a group $\longrightarrow$ $\mathbb{C}\langle \pi_1 X \rangle$ group ring

$\Gamma(B\pi_1; \mathrm{Vect}) = \mathrm{Vect}^{\pi_1} = \mathrm{Rep}(\pi)$.

$Q_2(X)$: $Q_1(\Omega X) = \bigoplus_{p \in \pi_1 X} \mathbb{C}\langle \pi_2 X \rangle = \bigoplus_{p \in \pi_1 X} \mathrm{Rep}(\pi_2 X)$

Hey it's a group $\Rightarrow$ new multiplication

convolution on components

Graded category with τ

$\mathrm{Rep}\, \pi_2 X = \mathrm{Vect}[\pi_2 X^\vee]$, pt wise.

$\pi_1 \ltimes \mathrm{Vect}[\pi_2 X^\vee]^\otimes$

$Q_3(X)$: $\pi_1 \ltimes \Big($ Braided fusion cat from π_2, π_3 $\Big)$

$\mathbb{C}\langle \pi_2 \rangle \langle \pi_2 \rangle, *$

k-invariant $H^4(B^2\pi_2; \pi_3)$

$\hookrightarrow$ $\pi_2 \to \pi_3$ quadratic

$K(\pi, n) \longrightarrow$ E_n object in $(m-n)$ categories

symmetric

π with conv.

Twisted examples:

$m=0$ $\tau \in H^1(X;\mathbb{C}^\times) \longrightarrow$ line bdle

Replace $\mathbb{C}[\pi_0 X]$ w/ flat sections

$m=1$ $\tau \in H^2(X;\mathbb{C}^\times) \longrightarrow$ proj rep

$m=2$ $\tau \in H^3(X;\mathbb{C}^\times)$ comes from π_1

$\pi_1^\tau \ltimes \mathrm{Vect}\,\pi_2 X^\vee]$

$m=3$ $H^4(X;\mathbb{C}^\times)$ contrib to braiding & DW for π_1..

Caution: π_n & higher do not contribute to Q_{n-1}

unless $\tau : \pi_n \to \mathbb{C}^\times$ is nontrivial

then 0.

π_{n-1} can be replaced by a collect of DW theories

(EM duality)

TQFT: $Q : \mathrm{Cor}_m \longrightarrow$ (m-category)

$M^k \longmapsto Q_{m-k-1}(\mathrm{Map}(M^k;X))$

II. EM duality – Strictly Commutative Case

Strictly commutative case: in dim n,

$$Q_{n-1}(\Sigma^p A) \equiv Q_{n-1}(\Sigma^{n-p-1} A^\vee)$$

by

$$\begin{tikzcd} & (\Sigma^p A \times \Sigma^{n-p-1} A^\vee, b) \arrow[dl] \arrow[dr] & \\ \Sigma^p A & & \Sigma^{n-p-1} A^\vee. \end{tikzcd}$$

Eg $n=1$, $p=0$: Fourier transform $\mathbb{C}[A] \cong \mathbb{C}[A^\vee]$

$n=2$, $p=1$: $BA \longleftrightarrow A^\vee$

$(\mathbb{C}\langle A\rangle\text{-Mod}) \longleftrightarrow \mathrm{Vect}[A^\vee]$.

$n=3$, $p=1$: Gauge theories for A and $A^\vee$ interchanged

$$(\mathrm{Rep}(A^\vee), \otimes =)\ \mathrm{Vect}\langle A\rangle \ \underset{\mathrm{Vect}}{\equiv}\ \mathrm{Vect}\langle A^\vee\rangle\ (= \mathrm{Rep}(A), \otimes)$$

Vect: trivial as left & right A, $A^\vee$ module sep

NONTRIVIAL as $A \times A^\vee$ bimodule

[Heisenberg extension]

$$\mathrm{Vect}\langle A\rangle \overset{\otimes}{=} \mathrm{End}_{\mathrm{Vect}\langle A^\vee\rangle}(\mathrm{Vect}).$$

Rmk this holds in nonab. case with $\mathrm{Vect}\langle G\rangle$, $\mathrm{Rep}(G)$

In general: $Q_{n-1}(\Sigma^{n-p-1} A^\vee) = \mathrm{End}_{Q_{n-1}(\Sigma^p A)}(\mathrm{Vect})$

Examples in 3D

Vect$\langle G \rangle$ alg object $1_g \hookrightarrow$ reg. module

alg object $\bigoplus_g \mathbb{C}_g = a$

$$a \otimes a = \bigoplus_{g,h} \mathbb{C}_{gh} \xrightarrow[\text{sum}]{} \bigoplus_k \mathbb{C}_k$$

modules = G vect bdles on G, Vect, Neumann

$H \subset G$ $\bigoplus_{h \in H} \mathbb{C}_h$ convolution : modules = Vect$[G/H]$

generating object: trivial bdle

Ctral ext $\tilde{H}$: Vect$^\tau(G/H)$, trivial bdle still ok

Rep(G) alg object $\mathbb{1} \leftrightarrow$ reg module

alg object $\mathbb{C}[G]$ as left G-module, plain mult

$\hookrightarrow$ G equivariant vector bdles on G, Vect

alg object $\mathbb{C}[G/H]$ as left G-module, pointwise

$\to$ Rep(H). Generator: trivial rep.

Ctral extension $\tilde{H} \leftarrow$ Rep$(\tilde{H})$

Choose proj rep E of H, $G \times_H$ End(E) algebra

Exercise (1) Work out statements for twisting

(2) For abelian groups: (Rep(A), $\otimes$) = (Vect$(A^\vee)$, $*$)

Find correspondence betw. centrally extended subgroups!

EM duality for Spectra ← postponed to III

Orthogonal group action on category of spectra:
naturally trivialized;
Second trivialization of action = action on Id functor
= J-action of O on spectra
$\Rightarrow$ two different orientation theory structures
interchanged by EM duality ✓

Boundary conditions on spectra:

$$F \hookrightarrow \boxed{Y \longrightarrow X} \qquad \boxed{\Sigma^{n-2} F^\vee \longrightarrow \Sigma^{n-1} X^\vee} \longrightarrow \Sigma^{n-1} Y^\vee$$

Dirichlet: $Y = X \Leftrightarrow F = \Sigma^{-1} X \Leftrightarrow \Sigma^{n-2} F^\vee = \Sigma^{n-1} X^\vee$ Neumann

Caution with connectivity:

$Y \longrightarrow X$ "Dirichlet" $\Leftrightarrow$ (generating)

$\pi_0 Y \to \pi_0 X$ onto
$\pi_{n-1} Y \to \pi_{n-1} X$ zero

$\Updownarrow$

$(\pi_{n-1} X)^\vee \longrightarrow (\pi_{n-1} Y)^\vee$ zero
$(\pi_0 X)^\vee \longrightarrow (\pi_0 Y)^\vee$ into

$\Updownarrow$

$\pi_0 \Sigma^{n-2} F^\vee \longrightarrow \pi_0 \Sigma^{n-1} X^\vee$ onto
$\pi_{n-1} \Sigma^{n-2} F^\vee \longrightarrow \pi_{n-1} \Sigma^{n-2} X^\vee$ zero

<u>Thm</u> Every $Y \to X$ satisfying is a 'generating' boundary condition: the algebra object

$$Q_n(Y \underset{X}{\times} Y) \text{ is } \equiv Q_n(X).$$

<u>Ex</u> explain Thm condition.

III. Recap, Complements & Examples

Representations. Duality for spectra reinterpreted.

Boundary conditions, duality & symmetries

$\Sigma^p A \longleftrightarrow \Sigma^{d-p-1} A^\vee$ give equivalent TQFTs

$\mathcal{Q}_{d-1}$: algebra objects in a $(d-1)$ category, Morita $\equiv$.

More general :

$A' \hookrightarrow A \to A''$, $e \in \mathrm{Ext}^1(A'', A') \Longleftrightarrow b : A'' \times (A')^\vee \to \Sigma \mathbb{C}^\times$

$\updownarrow$

$\Sigma^p A \Longleftrightarrow \Sigma^p A'' \times \Sigma^{d-p-1}(A')^\vee$, $\quad \tau : \Sigma^p A'' \times \Sigma^{d-p-1}(A')^\vee \to \Sigma^d \mathbb{C}^\times$

equivalent TQFTs

Spaces X : $d = \begin{matrix} 2n \\ 2n-1 \end{matrix}$ $\qquad X_{\geq n} \hookrightarrow X \twoheadrightarrow X_{<n}$

$\uparrow$

∞ loop space, except possibly when n even

and a quadratic $k^{2n} \in H^{2n}(\Sigma^n \pi_n ; \pi_{2n-1})$

$\to$ Requires SPECIAL HANDLING when n even:

$\Sigma^n \pi_n$, τ nondeg. quadratic to $\mathbb{C}^\times$

gives invertible TQFTs. Can be formally finalized.

This defines a new class of "self-dual theories"

in dim $2n-1$ which have no top. ∂ conditions.

Assuming that taken care of,

flip to $X_{<n} \times \Sigma^{d-1}(X_{\geq n}^\vee)$, τ $\qquad$ τ valued in $\Sigma^d I_{\mathbb{C}^\times}$

linear on Σ^{d-1}

from k-invariant $[X_{<n}, \Sigma X_{\geq n}]$

$n=4$: Partial duality : 2-group $\longrightarrow$ pure gauge theory

2-group: $B^2A \hookrightarrow X \twoheadrightarrow BG$ $\qquad \pi_2 = A$, $\pi_1 = G$

- G action on A
- $k \in H^3(BG;A)$
- $\tau \in H^4(X;\mathbb{C}^\times)$

$H^4(B^2A;\mathbb{C}^\times)$ $\leftarrow$ Quadratic part: assume 0 (else an invertible factor)

$$\begin{tikzcd}
H^4(B^2A;\mathbb{C}^\times) \arrow[rrrdd] & & & & & \\
0 & 0 & 0 & 0 & & \\
(A^\vee)^G \arrow[rrrdd] & H^1_G(A^\vee) \arrow[rrrdd] & \overset{e\,\in}{H^2_G(A^\vee)} \arrow[rrrdd, "k"] & H^3(BG;A^\vee) & 0 & \\
0 & 0 & 0 & 0 & 0 & \\
\mathbb{C}^\times & \widehat{G} & H^2(BG;\mathbb{C}^\times) & H^3(BG;\mathbb{C}^\times) & H^4(BG;\mathbb{C}^\times) & {}
\end{tikzcd}$$

Partial EM dual:

extension $1 \to A^\vee \hookrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$

$e \in H^2_G(A^\vee)$

$\tau' \in H^4_{\tilde G}(\mathbb{C}^\times)$ $\quad \left\{ \begin{array}{l} k \in H^3(BG; \overbrace{H^1(BA^\vee;\mathbb{C}^\times)}^{A}) \\ H^4(BG;\mathbb{C}^\times) \text{ from } \tau \end{array} \right.$

Exercise:

EM duality induced by correspondence with suitable τ'

$$\begin{tikzcd}
 & X \times_{BG} B\tilde{G} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
(X,\tau) & & (B\tilde{G},\tau')
\end{tikzcd}$$

$\leftarrow$ $p_2^*\tau' - p_1^*\tau$ trivializable

Representations of $\mathcal{Q}_{d-1}(X)$

On objects in the same $(d-1)$ category:

Formally, Local systems of such objects over X

= Fixed points (invariants) of ΩX on said $(d-1)$ category

= "representation of ΩX" in same

$\mathcal{Q}_{d-1}(X) \longrightarrow$ Reps is meant to be a form of "idempotent completion".

Commutative Case, $X = \Sigma^p A$: $\Sigma^{d-p-1} A^\vee$ is the "moduli space" of strictly commutative reps of $\Sigma^{p-1} A$ on Vect as a $(d-2)$ category $(\Sigma^{d-3}\mathrm{Vect})$

if $p < d-1$, a unique rep with automorphism gp $\Sigma^{d-p-2} A^\vee$

from bicharacter $\Sigma^{p-1} A \times \Sigma^{d-p-2} A^\vee \longrightarrow \Sigma^{d-1} \mathbb{C}^\times$

= central extension of $\mathcal{I}$ by $\Sigma^{d-1}\mathbb{C}^\times = \mathrm{Aut}\,\mathrm{Id}\,(\mathrm{Vect}$ as $(d-2)$ cat$)$

Problem Action of $\Sigma^{p-1} A$ on $\Sigma^{d-3}\mathrm{Vect}$ up to iso

$\longleftrightarrow \quad H^{d-1}(\Sigma^{p-1}A; \mathbb{C}^\times)$

these are components of the moduli space of reps

MISMATCH!

Reason: Mixing metaphors.

Strictly abelian world = chain complexes

$\longrightarrow$ moduli space is $R\mathrm{Hom}(\Sigma^{p-1}A, \Sigma^{d-1}\mathbb{C}^\times) = \Sigma^{d-p-2}A^\vee$.

Cohomology comes from the homotopical world,
replacement for injective module $\mathbb{Q}/\mathbb{Z}$ ($\mathbb{C}^\times$) is $I_{\mathbb{C}^\times}$
(Linear) replacement for RHom is

$$\mathrm{Stable\ Map}(\mathcal{X}; I_{\mathbb{C}^\times}) = \mathcal{X}^\vee$$

$\hookrightarrow$ infinite loop spaces (E_∞ groups)

So $\Sigma^{d-p-1}A^\vee$ is the moduli of E_∞ 1-dim reps of $\Sigma^{p-1}A$ on $\Sigma^{d-1}\mathbb{C}$

- Not true that all 1-dim reps are E_∞ (true characters)
- Not true that every rep splits into 1-dims.

Eg $d=5$, $X = B^2A$, reps on $\Sigma^4\mathbb{C}$ $\quad A = \mathbb{Z}/2$

$H^4(B^2A; \mathbb{C}^\times) = \mathbb{Z}/4$ (quadratic forms)

braided tensor 2-category (braiding binial)

4 BTCs based on A

each is a "twisted Neumann" bdy condition

$\mathbb{Z}/2 \subset \mathbb{Z}/4$ is <u>stable</u> ($i \mathbb{S}_{\mathbb{Z}}^2$)

Conhadwicks classification?

No! killed after inclusion $\mathbb{C}^\times \hookrightarrow \mathbb{C}^\times_{\mathbb{Z}/2}$

(superVector spaces)

Interesting representation: 4d gauge theory BA (with DW twist)

Not a sum of 1d TQFTs.

Representations as boundary conditions

Bdry conditions: interfaces with trivial theory

$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr] & \\ X & & * \end{tikzcd}$$

Associated local system on X: $R\pi_*(\Sigma^{d-1}\mathbb{C})$

Fiber = $(d-1)$-quantization of fiber F

– has an ΩX action (in $\Sigma^{d-1} I_{\mathbb{C}}^{\times}$)

$Y = X$, possibly with a cocycle: a Neumann cond

$Y =$ a point in each component:

$F = \left(\coprod_{\pi_0}\right) \Omega_p X$ – with $\Omega_p X$ transl. action

"Dirichlet ∂ condition"

Note $\langle \text{Dirichlet} \,|\, TQFT_d \,|\, \text{Neumann} \rangle = \mathbb{1}$.

Can characterize Neumann conds in terms of Dirichlet.

Q What characterizes Dirichlet?

A: $\mathrm{End}_{\mathcal{Q}_{d-1}(X)}(D) \simeq \mathcal{Q}_{d-1}(X)$ Morita, via D.

Thm (Ostrik, Etingof et al) For an indecomposable fusion cat, every non-zero module cat is Dirichlet.

Thm $Y \to X$ is Dirichlet iff:

* $\pi_0 Y \to \pi_0 X$ is onto
* $\pi_{d-1} Y \to \pi_{d-1} X$ is ZERO.

Eg: $BG \to BG$ NOT Dirichlet in dim 2.

EM duality and ∂ conditions

Orthogonal group action on category of spectra: naturally trivialized;

Second trivialization of action = action on Id functor

= J-action of O on spectra

$\Longrightarrow$ two different oriented theory structures interchanged by EM duality ✓

Boundary conditions on spectra:

$$F \hookrightarrow \boxed{Y \longrightarrow X} \qquad \boxed{\Sigma^{n-2} F^{\vee} \longrightarrow \Sigma^{n-1} X^{\vee}} \longrightarrow \Sigma^{n-1} Y^{\vee}$$

Dirichlet: $Y = X \Longleftrightarrow F = \Sigma^{-1} X \Longleftrightarrow \Sigma^{n-2} F^{\vee} = \Sigma^{n-1} X^{\vee}$ Neumann

Caution with connectivity!

$Y \longrightarrow X$ "Dirichlet" (generating) $\Longleftrightarrow$ $\pi_0 Y \to \pi_0 X$ onto, $\pi_{n-1} Y \to \pi_{n-1} X$ zero

$\Updownarrow$

$(\pi_{n-1} X)^{\vee} \longrightarrow (\pi_{n-1} Y)^{\vee}$ zero

$(\pi_0 X)^{\vee} \longrightarrow (\pi_0 Y)^{\vee}$ into

$\Updownarrow$

$\pi_0 \Sigma^{n-2} F^{\vee} \longrightarrow \pi_0 \Sigma^{n-1} X^{\vee}$ ONTO

$\pi_{n-1} \Sigma^{n-2} F^{\vee} \longrightarrow \pi_{n-1} \Sigma^{n-1} X^{\vee}$ ZERO

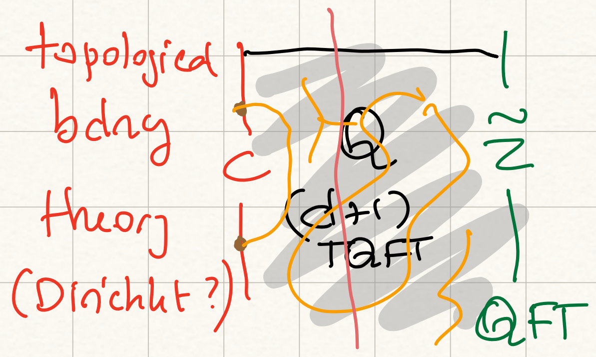
Thm Every $Y \to X$ satisfying is a 'generating' boundary condition: the algebra object

$$Q_{n-1}(Y \underset{X}{\times} Y) \text{ is } \equiv Q_{n-1}(X).$$

Ex explain this condition.

Symmetries

Def A (generalized) SYMMETRY of a QFT Z is:



$\longrightarrow Z_{QFT}$

Isomorphism of reduced theory with Z

This should be viewed as an action of $\mathrm{End}_{Q}(C)$ on Z

Extended topological operators in Z are the projections of topological defects in (C, Q) (not touching $\tilde{Z}$ boundary)

Those not touching C are "central" in a sense (below)

Remarks (1) "(-1) form symmetries" = self-domain walls of Q.
They change C. "Half-space" gauging can be done that way.
(2) "0-form symmetries" = self-defects of C.

(3) All interior operators may be pushed to the ∂.
(but may acquire singular support)
(4) D_p, D_q may "braid" interestingly if they link in the ∂.
But if one comes from the interior, braiding = trivial.
(5) More generally, this is true (in bulk or ∂)
if one of the defects is topologically condensed

$Q_{cl}(x)$ case: Braiding is ~~nontrivial~~ nondegenerate on defects with full support.

Calculus of Defects.

Internal to $\mathcal{Q}$:

linking sphere S^ℓ (bundle over the defect)

support of defect of codim $(\ell+1)$ in $(d+1)$

(Quantum) defect local label:

Boundary theory for $\mathcal{Q}(S^\ell)$ ($(d+1)-\ell$ dim TQFT)

If $\mathcal{Q}$ comes from X, this is a representation on $\Sigma^{d-\ell}\mathbb{C}^\times$ of $\mathcal{Q}_{d-\ell}(\mathrm{Map}(S^\ell;X))$

Caution: The spheres may rotate along the defect (w/ normal bdle)

- Even on oriented theories (eg X with $\tau \in H^{d+1}(X;\mathbb{C}^\times)$) the $SO(\ell+1)$ action on ⋯ may be nontrivial. (Source: k-invariants of X)

We need a "flat" or locally constant ∂ theory for a choice of quantum label all along defect

- Constraints on tangential structures are tricky to work out.

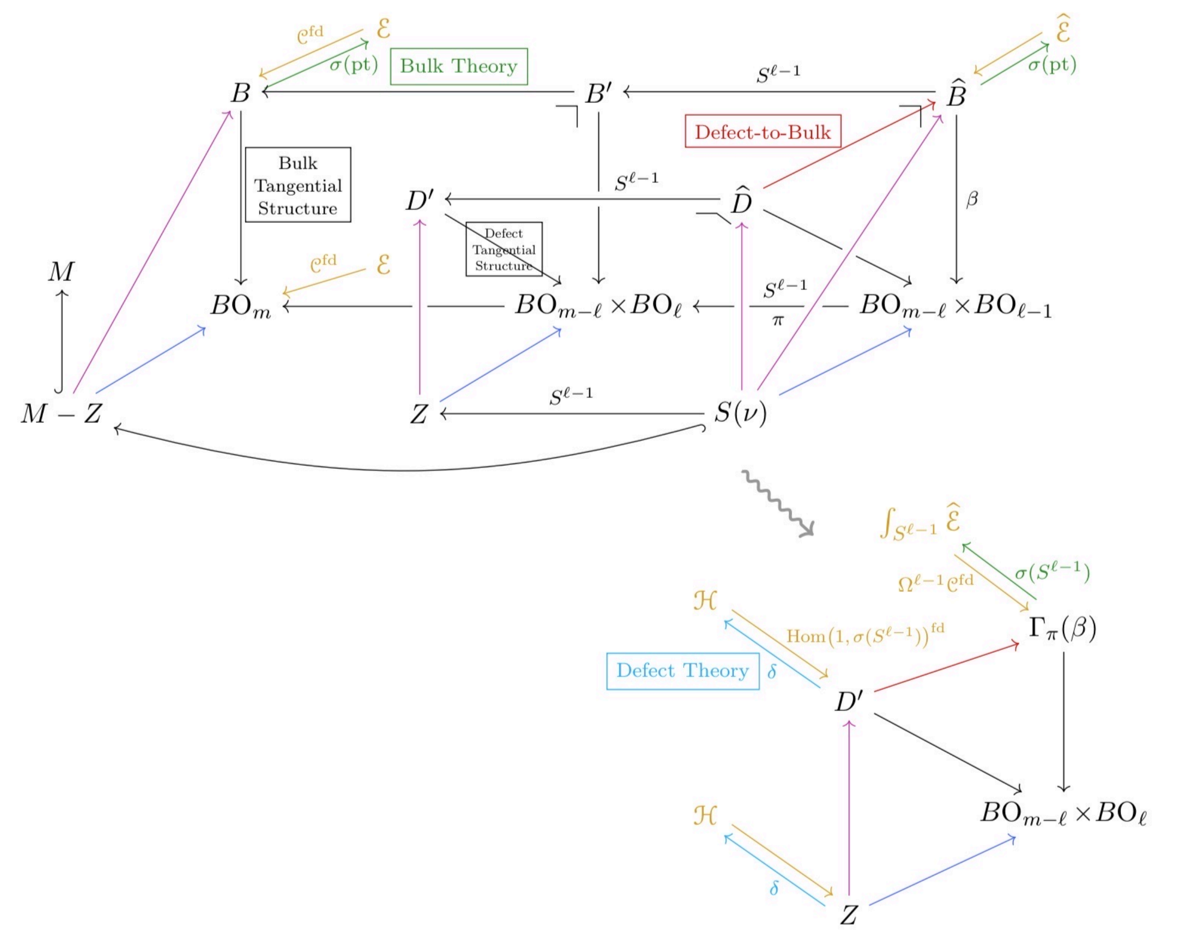


FIGURE 8. Local defect data, including tangential structures

Defects in an n-dimensional TQFT from a space X

$\mathcal{Q}_m(Y) :=$ quantization as algebra in an m-category

Top row = constant X = "Wilson defects"

Bottom row = $\Omega^{\cdot} X$ = 't Hooft defects

Codimension of defects	Local quantum labels are modules over…	Contributing homotopy groups, placed in dim [d] all/$\boxed{\text{new}}$ defects	Condensed
n (=points)	$\mathcal{Q}_0(X^{S^{n-1}})$	$\boxed{\pi_0}$ Wilson $\boxed{\pi_{n-1}[0]}$ 't Hooft	None
n−1 (lines)	$\mathcal{Q}_1(X^{S^{n-2}})$	$\pi_0, \boxed{\pi_1}$ $\boxed{\pi_{n-2}[0]}, \pi_{n-1}[1]$	π_0 $\pi_{n-1}[1]$
n−2 (surfaces)	$\mathcal{Q}_2(X^{S^{n-3}})$	$\pi_0, \pi_1, \boxed{\pi_2}$ $\boxed{\pi_{n-3}[0]}, \pi_{n-2}[1], \pi_{n-1}[2]$	π_0, π_1 $\pi_{n-2}[1], \pi_{n-1}[2]$
$k = n - \lfloor \frac{n-1}{2} \rfloor$	$\mathcal{Q}_{n-k}(X^{S^{k-1}})$	$\pi_0, \pi_1, \dots, \boxed{\pi_{\lfloor \frac{n-1}{2} \rfloor}}$ $\boxed{\pi_{\lceil \frac{n-1}{2} \rceil}[0]}, \pi_k[1], \pi_{k+1}[2], \dots$	$\pi_0, \pi_1, \dots, \pi_{\lfloor \frac{n-1}{2} \rfloor - 1}$ $\pi_k[1], \pi_{k+1}[2], \dots$
$k = n - \lfloor \frac{n-1}{2} \rfloor - 1$		$\pi_0, \pi_1, \dots, \boxed{\pi_{\lfloor \frac{n-1}{2} \rfloor + 1}}$ $\boxed{\pi_{k-1}[0]}, \pi_k[1], \pi_{k+1}[2], \dots$	$\pi_0, \pi_1, \dots,$ $\pi_k[1], \pi_{k+1}[2],$
2	$\mathcal{Q}_{n-2}(X^{S^1})$	$\pi_0, \pi_1, \dots, \boxed{\pi_{n-2}}$ $\boxed{\pi_1[0]}, \pi_2[1] \dots, \pi_{n-1}[n-2]$	$\pi_0, \pi_1, \dots,$ $\pi_2[1] \dots, \pi_{n-1}[n-2]$
1 (interface)	$\mathcal{Q}_{n-1}(X^{S^0})$	$\left[\begin{matrix} \pi_0, \pi_1, \dots, \pi_{n-1} \\ \pi_0[0], \pi_1[1], \dots \pi_{n-1}[n-1] \end{matrix} \right]$	$\Delta\pi_0$ $\pi_1, \dots \pi_{n-2},$ $\Delta\pi_{n-1}$ $\pi_1[1], \pi_2[2] \dots,$
0 (space-filling)	$\mathcal{Q}_n(X^{S^{-1}})$	$\pi_0, \pi_1, \dots, \boxed{\pi_n}$ $\boxed{\pi_n[0]}, \pi_0[1], \pi_1[2] \dots$	$\pi_0, \pi_1, \dots \pi_{n-1}$ $\pi_0[1], \pi_1[2] \dots$

- DW cocycle $\tau \in I^n_{\mathbb{C}^\times}(X)$ can be integrated over S^k

π_1-ganging of 't Hooft defects needed if π_1 acts nontrivially

1-fold Condensation

Work in progress w Freed, Hopkins;
(multifold condensation, singular.. in progress)

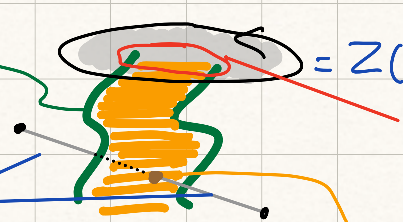
Key ingredients

1. Notion of Dirichlet boundary condition for defects
   [Higher generalization with handles]
2. Correspondence between internal algebra objects in a tensor category & module categories
   [Ostrik for fusion cats; await full generalization]

Remark It is clear how this works if we work in the category of iterated algebras or categories.
But extra wishes on the list need more care.

Example (3d, connected theory, fusion cat F)
(1) All condensed line operators are scalars
(2) All surface operators are condensed

Pf (1) Only scalar point ops (connectedness)
(2) surface ops $\longleftrightarrow$ $F \boxtimes F^{mo}$-module cats
$\longleftrightarrow$ $Z(F)$-module cats $\longleftrightarrow$ algebra objects in $Z(F)$
up to Morita equiv.
fixed by the end



algebra object $a \in Z(F)$

condensate F-F bimodule category ${}_F M_F$

bimodule functor ${}_F F_F \xleftarrow{\text{end}} {}_F M_F$

Precise Condition Observe the adjunction (TQFT $\mathcal{J}$)

$$\mathrm{Hom}_{\Omega^{\ell+1}\mathcal{C}}(\mathbb{1}_{\ell+1}, \mathcal{J}(S^{\ell+1})) \cong \mathrm{End}_{\mathrm{Hom}(\mathbb{1}_\ell, \mathcal{J}(S^\ell))}(\mathcal{J}(D^{\ell+1}))$$

$$\begin{tikzcd} \mathbb{1}_\ell \arrow[r, "\mathcal{J}(D^{\ell+1})"] & \mathcal{J}(S^\ell) \arrow[l, bend left, "\mathcal{J}(D^{\ell+1})^*"] \end{tikzcd}$$

$$\mathcal{J}(D^{\ell+1})^* \circ_{\mathcal{J}(S^\ell)} \mathcal{J}(D^{\ell+1})$$

$\cap$

$$\Omega^\ell \mathcal{C} := \underbrace{\mathrm{End}(\mathrm{End}(\ldots(\mathrm{End}}_{\ell \text{ times}}(\mathbb{1}_0))\ldots)$$

$\mathcal{C}$ ↑ target N-cat.

(Ostrik principle)

(nice) algebra objects in LHS = modules over RHS (with a generator)

(Localization assumption)

Objects of $\mathrm{Hom}(\mathbb{1}_\ell, \mathcal{J}(S^\ell))$ are "determined by" their "localization at the unit" $\mathcal{J}(D^{\ell+1})$

"Localization at $\mathcal{J}(D^{\ell+1})$ is an equivalence"

$$\boxed{\begin{array}{l} M \in \mathrm{Hom}_{\Omega^\ell \mathcal{C}}(\mathbb{1}_\ell, \mathcal{J}(S^\ell)) \\ + \ \mathrm{end} \in \mathrm{Hom}(\mathcal{J}(D^{\ell+1}), M) \end{array}} \xrightarrow{\text{localize}} \mathrm{Hom}_{\mathrm{Hom}_{\Omega^\ell\mathcal{C}}(\mathbb{1}_\ell, \mathcal{J}(S^\ell))}(\mathcal{J}(D^{\ell+1}), M),$$

a module over

$$\mathrm{End}_{\mathrm{Hom}(\mathbb{1}_\ell, \mathcal{J}(S^\ell))}(\mathcal{J}(D^{\ell+1}))$$

+ generator

$$\boxed{\begin{array}{c} \text{algebra object} \\ a \in \mathrm{Hom}(\mathbb{1}, \mathcal{J}(S^{\ell+1})) \\ + \text{ regular module} \end{array}} \equiv \xleftarrow{\text{Ostrik principle}}$$

Dirichlet condition on end says that you can "condense back" to the original (defect, end).

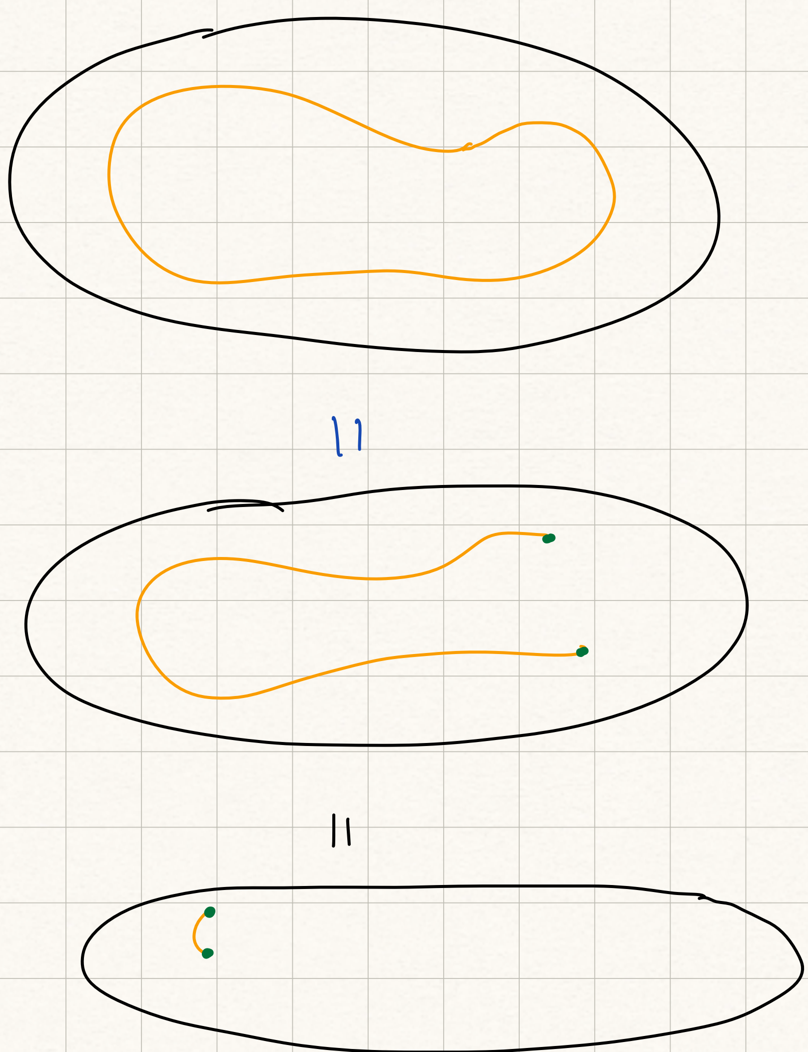
Overall

$\mathcal{A}$ = algebra in codim-$(\ell+2)$ defect labels $\mathrm{Hom}(\mathbb{1}_{\ell+1}, \Theta_d(X)[S^{\ell+1}])$
+ regular module (condenser)

$\longrightarrow \mathrm{End}_{\ldots}[\mathcal{J}(D^{\ell+1})]$ module (+ generator)

$\longrightarrow$ codim$(\ell+1)$ defect label in $\mathrm{Hom}(\mathbb{1}_\ell, \Theta_d(X)[S^\ell])$
(+ end) (condensate)

"Ripping open theorem" [Weak assumption on Id. completion]



A top-dimensionally integrated 1-fold condensed defect may be punctured and reattached to one codim lower to produce the same state vector in a given enclosing hypersurface,

at the cost of embedding additional defects in the boundary.

$Q_d(X)[S^{\ell+1}] \in (d-\ell-1)$ algebras //

& label $\in \mathrm{Hom}_{\Sigma^{d-\ell}\mathcal{C}}(\mathbb{1}, Q_d(X)[S^{\ell+1}])$ in part a $(d-\ell-1)$ alg

if algebra object then a $(d-\ell)$ algebra

$$\mathrm{Hom}_{\Sigma^{d-\ell+1}\mathcal{C}}(\mathbb{1}, Q_d(X)[S^{\ell}])$$

$$Q_d(X)[S^{\ell+1}] = U \otimes_{Q_d(X)[S^{\ell}]} U^{\vee}$$

$$\mathrm{Hom}(\mathbb{1}, Q_d(X)[S^{\ell+1}]) = \Omega\, \mathrm{Hom}(\mathbb{1}, Q_d(X)[S^{\ell}])$$

$\cup$

δ algebra object $\longmapsto$ module on object in $\mathrm{Hom}(\mathbb{1}, Q_d(X)[S^{\ell}])$ with end

adjunction:

$$\mathrm{Hom}(\mathbb{1}, Z(S^{\ell+1})) = \mathrm{End}_{\mathrm{Hom}(\mathbb{1}, Z(S^{\ell}))}(Z(D^{\ell+1}))$$

$\cup$

δ algebra $\longmapsto$ module on

unit for algebra

$$1 \xrightarrow{Z(D^{\ell+1})} Z(S^{\ell})$$

Same as object in $\mathrm{Hom}(\mathbb{1}, Z(S^{\ell}))$

$\Omega^{\ell}\mathcal{C}$ algebra object