1. 3D Topological gauge theory

The past decade led to an understanding that

3D topological gauge theory (A-model)
is controlled by
Hyperkähler spaces closely related to the Toda system

Key steps along the way

- Seiberg & Witten on 3D gauge theory for $SU(2)$
- Argyres-Faraggi et al for $SU(n)$
- Seiberg & Intriligator on 3D mirror symmetry
- Witten-Hanany on Coulomb branches
- Bezrukavnikov, Finkelberg, Mirkovic
  Topological construction of Toda spaces from affine Grassmannians
- Your distinguished speaker on Gromov-Witten boundary condns.
- Bullimore-Dimofte-Gaiotto abelian Coulomb branches
- Braverman-Finkelberg-Nakajima polarized matter
- Braverman et al proposal for quaternionic matter
- (–) Construction —"—

Template: QFT $\longrightarrow$ moduli space of vacua $\mathcal{M}$
controls low energy regime of theory

In SUSY gauge theories: Coulomb and Higgs branches of $\mathcal{M}$

3D: X hyperkähler with G action,

Higgs: $X/G_{\mathbb{H}}$ Coulomb: Toda (G) + "quantum corrections"

2. Refresher on the Toda spaces $\mathcal{C}_{3,4}(G;\mathcal{O})$

representation X

classical, H_* — periodic, K_*

- Hyperkähler manifolds (orbifolds) with completely integrable, abelian group structures over

$$\mathcal{C}_3 \longrightarrow \mathfrak{g}_{\mathbb{C}}/\!/G_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}}/W, \quad \mathcal{C}_4 \longrightarrow G_{\mathbb{C}}/\!/G_{\mathbb{C}} = T_{\mathbb{C}}/W$$

- Abelian cases: $T^*T_{\mathbb{C}}^\vee, \quad T_{\mathbb{C}} \times T_{\mathbb{C}}^\vee$

HK structure I

Nahm solution space

- General: (affine) blow-ups of Weyl quotients

- $\mathcal{C}_3(G;0) = H_*^G(\Omega G)$, $\quad G\backslash\Omega G = G\backslash LG/G$
  $\mathcal{C}_4(G;0) = K_*^G(\Omega G)$, $\quad$ Pontryagin product

  (BFM)

  Hopf algebras over H_*^G, K_*^G

- Boundary conditions for the 3D gauge theory come from symplectic manifolds w/ Hamilt. G-action. They define Lagrangians in the Toda spaces

Eg: representation V, Lagrangians are graphs of

$$\mathfrak{t}_{\mathbb{C}} \ni \xi \longmapsto \prod_{\nu} \left(\mu + \langle \nu|\xi\rangle\right)^{\nu} \in T_{\mathbb{C}}^\vee$$

$$T_{\mathbb{C}} \ni x \longmapsto \prod_{\nu} \left(1 - m^{-1}x^{-\nu}\right)^{\nu} \in T_{\mathbb{C}}^\vee$$

μ, m: "mass parameters"

$\exp(dW)$

GLSM superpotentials for V/G

3. Main Theorem on Coulomb branches $\mathcal{C}_{3,4}(G;E)$

G = compact connected Lie gp; E = quaternionic rep;
"polarized" means $E = V \oplus V^*$

Nakajima; Bullimore–Dimofte–Gaiotto; yours truly;
Braverman–Finkelberg–Nakajima; Braverman et al;

1. There exist[4] constructible, equivariant coefficient systems $\mathcal{H}_E$, $\mathcal{K}_E$ over the loop Grassmannian $G_{\mathbb{C}}[[z]] \backslash G_{\mathbb{C}}((z)) / G_{\mathbb{C}}[[z]]$
homotopy $\simeq$ equiv $G \backslash \Omega G$

2. They are E_2-multiplicative under Pontryagin products and their equivariant cohomologies $\mathcal{C}_{3,4}(G;E)$ are E_3 ("Poisson structures of degree -2")

3. They are multiplicative in E, $\mathcal{H}_E \otimes \mathcal{H}_F \to \mathcal{H}_{E \oplus F}$
so $\mathcal{C}_{3,4}(G;E) \underset{\text{Toda}}{\times} \mathcal{C}_{3,4}(G;F) \longrightarrow \mathcal{C}_{3,4}(G;E \oplus F)$

4. Non-polarized E require the removal of obstructions.

5. $H_*^G(\Omega G; \mathcal{H}_E)$ and $K_*^G(\Omega G; \mathcal{K}_E)$ are birational to $\mathcal{C}_3$, $\mathcal{C}_4$ and are expected to be the Coulomb branches for E/G

6. (Abelianization) $\mathcal{C}_{3,4}(G;E) \cong \mathcal{C}_{3,4}(T; E - \mathfrak{g}_{\mathbb{H}})/W$
if E contains the roots of $\mathfrak{g}$. [–]

7. Polarized case: construction from GLSM boundary cond.
[–]

4. Construction of Coulomb branches – polarized case

(Physics; Nakajima; B-F-N; B-D-G)

Moral construction: $E = V \oplus V^{\vee}$, choose one of them

Get an index bundle "$H^0 - H^1$" $(\mathbb{P}^1; P \times_G V)$ along $\mathbb{P}^1$ $\longrightarrow \sqrt{K}$ tweak

over the moduli $\mathrm{Bun}_G(\mathbb{P}^1) \sim G\backslash \Omega G$ of holomorphic $G_{\mathbb{C}}$-bundles on $\mathbb{P}^1$.

Build the linear space $L_V := \mathrm{Spec}\,\mathrm{Sym}$ of its (check) index sheaf

Get coefficient systems $\mathcal{H}_E, \mathcal{K}_E$ by forming cohomology with compact vertical supports

$\longrightarrow$ 'shifts' the strata of $\mathrm{Bun}_G(\mathbb{P}^1)$ by fiberwise Euler classes

Morally, $\mathcal{C}_{3,4}(G;E) := \mathrm{Spec}\, H^G_*(\Omega G; \mathcal{H}_E, \mathcal{K}_E)$.

Product structure is expected to come from the 3D pair of pants by solving the Dirac equation with prescribed boundary conditions:

$$\begin{tikzcd} & (\text{Mod space of solutions}) \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G(\mathbb{P}^1) \times \mathrm{Bun}_G(\mathbb{P}^1) & & \mathrm{Bun}_G(\mathbb{P}^1) \end{tikzcd}$$

$\longrightarrow$ (in E)

5. Algebraic geometry rewording (BFN)

The analysis for the respective Dirac equation is not complete, but an alternative developed by BFN exploits the $\mathbb{R}\times\mathbb{C}$ splitting of $\mathbb{R}^3$ to reduce the Dirac eq. to $\bar\partial$ and hence to complex geometry

(See paper for details) but use "tiny sphere" instead of $\mathbb{P}^1$

$= -:-$ two copies of the disk glued away from 0

Moduli of $G_\mathbb{C}$-bundles $= G_\mathbb{C}[[z]]\backslash G_\mathbb{C}((z))/G_\mathbb{C}[[z]]$

has the same homotopy type $G\backslash\Omega G$ as $\mathrm{Bun}_G(\mathbb{P}^1)$

Multiplication can be defined by Hecke correspondences.

6. Theorem (Global structure, polarized case)

$\mathcal{C}_{3,4}(G; V\oplus V^*)$ is obtained by gluing two copies of $\mathcal{C}_{3,4}(G;0)$ sheared by the Lagrangian shift $\exp(dW)$ for the GLSM potential W.

Reformulate: $\mathcal{C}_{3,4}(G;V\oplus V^*) \longrightarrow$ (Toda base $\mathfrak{t}, T/W$) has two Lagrangian sections, from V, V^*, whose ratio is the said Lagrangian shift; and it is covered by the two Toda charts defined by these sections.

("quasi-toric calculus for Toda group scheme")

7. Non-polarized case

Invoking the earlier construction for E instead of V leads to the 'doubled' Coulomb branches $\mathcal{C}_{3,4}(G; E \oplus E)$.

So the problem is to extract square roots of $\mathcal{H}_{E\oplus E}$, $\mathcal{K}_{E\oplus E}$ that is, of fiberwise Euler classes

A way to cut a complex space in half: by a real structure.

Investigate:

$$\begin{tikzcd}
 & & BU \arrow[dl, "\text{doubling}"'] & V \arrow[d, mapsto] & & & KO \subset KU \arrow[d] & \\
 & BG \arrow[r, "E"] & BSp \arrow[r, squiggly] & V \oplus V^* & \Omega G \arrow[r, "\Omega^2 E", "G\text{-map}"'] & \Sigma^2 KO \arrow[r, "\eta"] & \Sigma KO &
\end{tikzcd}$$

A polarization of E would give a lift of $\Omega^2 E$ to KU

Obstructed by $\eta \circ \Omega^2 E \in KO^1$.

In any case we'd want an E_2 lift so obstruction really is

$$BG \xrightarrow{E} BSp \xrightarrow{\eta} \Sigma^3 KO$$

Seems unhelpful until we recall that

we don't need a complete lift!

Just enough to build the coefficient systems.

So the obstruction is the image, via $\Sigma^4 J$, into $\Sigma^4 GL_1(H\mathbb{Z})$ or $\Sigma^4 GL_1(KU)$ (or $\Sigma^4 GL_1(KO)$)

For cohomology : obstruction class in $H^4(BG;\mathbb{Z}/2)$
(W_1) and is $c_2(E) \bmod 2 = w_4(E)$

For KO-theory : a secondary obstruction $\sigma \in H^5(BG;\mathbb{Z}/2)$
(W_2) is defined if $w_4(E) = 0$

For KU-theory : the 2nd obstruction is $B\sigma \in H^6(BG;\mathbb{Z})$
(W_3) (Essentially $\frac{1}{2} c_3(E)$)

Theorem (nasty calculation)

If G is connected and $w_4(E) = 0$, then $B\sigma = 0$.

(Fails for disconnected groups)

Improvements · One can weaken the obstruction to
w_4 is the square of a class in $H^2(BG;\mathbb{Z})$

· One can even reduce to the obstruction predicted by
Ed Witten $\Longleftrightarrow$ w_4 has a square root $\in H^2(BG;\mathbb{Z}/2)$

at the price of collapsing
the cohomology grading mod 2:

$$BG \xrightarrow{E} BS_4 \xrightarrow{\eta} \begin{array}{cc} 0 & \\ \mathbb{Z}/2 & 5 \\ \mathbb{Z}/2 & 4 \\ \mathbb{Z} & 3 \\ 0 & \end{array}$$

homology grading $\longrightarrow$