Categorical representation and character theory

Constantin Teleman

UC Berkeley

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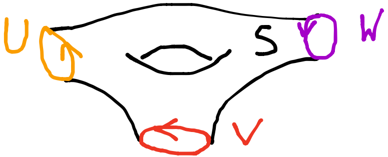
Representation theory = 2D topological gauge theory

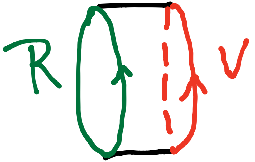
A 1-dimensional dimensional TQFT is a (finite-dimensional) vector space. The invariant for a circle is its dimension ("1st Hochschild homology").

Extra structure on the vector space can often be imposed by promoting it to a *relative* theory, a boundary condition for a 2D TQFT:



In G-gauge theory, the invariant of S is the sum ($\int$) of $Tr_U(O)\,Tr_V(O)\,Tr_W(O)$ over (flat) principal bundles on Σ.

The original TQFT reappears by reduction along the interval with the regular representation as the other boundary condition: $\mathrm{Hom}^G(R;V) = V$. The picture on the right computes $\dim V$.



Topological group actions on categories

Definition (Action of G on a (linear) category $\mathcal{C}$)

- An endofunctor Φ_g of $\mathcal{C}$ for each $g \in G$
- A natural isomorphism $\Phi_{gh} \xrightarrow{\sim} \Phi_g \circ \Phi_h$ for each pair,

with an obvious coherence condition for all triples (g, h, k).

Remark

1. This is a homomorphism from G to the 2-group of auto-functors of $\mathcal{C}$.
2. *Topological*: add continuity *and* a *trivialization* $\Phi \cong \mathrm{Id}$ near $1 \in G$.
3. Derived version: relax equalities to coherent homotopies.

Definition (invariant category $\mathcal{C}^G$)

- Objects: $\{\text{tuples } \{x; \varphi_{x,g} : x \xrightarrow{\sim} \Phi_g x\}_{g \in G} \mid \text{coherence condition}\}$
- Morphisms: $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ commuting with the φ.

Is there a character theory?

Seems hopeless at first: Schur's lemma fails.

- An endomorphism of $\mathcal{V}ect$ is (tensoring with) a vector space V. Equivariance for the trivial G-action amounts to a G-action on V. So, $\mathrm{End}^G(\mathcal{V}ect) =$ tensor category of G-representations.
- Similarly, a central extension $\mathbb{C}^\times \to G^\tau \to G$ gives a G-representation $\mathcal{V}ect^\tau$ on $\mathcal{V}ect$, and $\mathrm{Hom}^G\left(\mathcal{V}ect^1, \mathcal{V}ect^\tau\right) =$ category of projective G-representations. ("Homs between distinct irreducibles".)

No spectral theory of geometric flavor can account for this.
(No character theory for categorical representations of finite groups.)

Recall though: TQFT $\rightsquigarrow$ *boundary conditions for pure 3D gauge theory*. Connected G leads to a hyper-Kähler space of vacua.

This postdicts the shape of the character theory for connected G, which can be framed mathematically.

BFM space of vacua from the Langlands dual group $G^\vee$

Theorem (Bezrukavnikov-Finkelberg-Mirkovic + small improvements)

1. *Spec* $H_*^G(\Omega G)$ *is an algebraic symplectic manifold, isomorphic to the algebraic symplectic reduction* $T^*_{reg} G^\vee //_{Ad} G^\vee$.
2. *It is an affine resolution of singularities of* $(T^* T_{\mathbb{C}}^\vee)/W$.
3. *The fiber of Spec* $H_*^G(\Omega G)$ *over* $0 \subset \mathfrak{t}_{\mathbb{C}}/W$ *is a Lagrangian submanifold* $\cong$ *Spec* $H_*(\Omega G)$. *("Regular representation".)*
4. *Spec* $K_*^G(\Omega G)$ *is is an algebraic symplectic orbifold, isomorphic to a twisted holomorphic symplectic reduction* $T^*_{reg}(LG_{\mathbb{C}})^\vee //_{Ad} (LG_{\mathbb{C}})^\vee$.
5. *It is an affine (orbi-)resolution of singularities of* $(T_{\mathbb{C}} \times T_{\mathbb{C}}^\vee)/W$
6. *See (3), mutatis mutandis.*

G- (*LG*-)categories **localize over Lagrangian sub-varieties** on *BFM*. they (should represent boundary conditions for its Rozansky-Witten theory. *BFM*(G) is also a space of solutions to Nahm's equations, and carries a complete hyper-Kähler structure. Conjecturally true for (4).

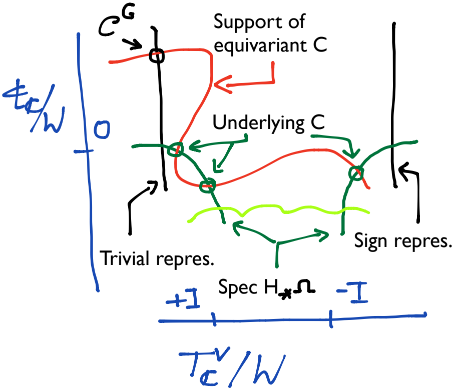
Theorem (After interpretation)

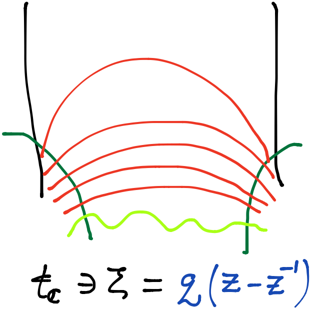
1. *Character theory of topological G-representations is controlled by the (holomorphic) Lagrangian geometry of $BFM(G^\vee)$.*
2. *$BFM(G^\vee)$ is foliated by complete family of irreducible G-categories coming from the Toda integrable system.*
3. *The G-representations 'are' the Gromov-Witten theories of G-flag varieties. ("Borel-Weil construction")*
4. *(Preliminary.) For the loop group LG, the folitation is constructed form the (twisted) Poisson deformations of the Toda systems of $G^\vee$.*
5. *The LG-representations are their Gromov-Witten K-theories of G-flag varieties.*

Remark

1. Much of the content is based on work of Givental and collaborators.
2. The naive generalization from G to LG fails: it gives instead the Gromov-Witten theories of flag varieties of LG (Mare-Mihalcea 2014).

Character space for SO(3) and Toda foliation





Open questions

1. **GIT quotient conjecture.** Fukaya categories of compact symplectic Hamiltonian G-manifolds are topological G-representations.
   Their fixed-point categories should be the Fukaya categories of the Hamiltonian reductions (computable by character calculus).
   More generally, the spectral decomposition of the Fukaya category should match the decomposition of the space under the moment map.
2. **Conjecture is false!** Without boundaries. Reducing outside the moment map image gives zero; character calculus can only replicate that if the Lagrangians have real boundaries. Intrinsic meaning of the boundary? (Stability conditions?)
3. K-**theory.** The cohomological story is equivariant mirror symmetry (done correctly). What is the role of K-theory? Gromov-Witten K-theory (fake) is GW with coefficients, but the G-version involves LG by a Chern character map, which appears to give a good representation theory for LG. Seems to want to be 4D gauge theory.

4. **Loop groups vs quantum groups** The 'nice' categorical loop group representations rely on the Poisson-Lie (or quantum) group. Explain? Note that another (more direct) relation between quantum group and loop group reps appear in another 3D gauge theory (Chern-Simons).
5. **Characterize the characters.** At present they are derived from Givental et al. calculation of the Gromov-Witten (K-) theories.
6. 3**D Mirror symmetry.** This seems closely related to mirror symmetry in $3D$ (Nakajima; Bullimore-Dimofte-Gaiotto). Is there a story with 'equivariant matter'? Do the topological boundary conditions for those relate to Gromov-Witten theory?
7. **B to A.** The other direction of equivariant gauge theory, mirror to group actions on algebraic varieties should involves a holomorphic Fukaya category of BFM and should use the hyper-Kähler structure. For a torus, the requisite 2-category is that of local systems of categories over $T^\vee$, by a categorified Fourier-Mukai transform.

Group actions and Hamiltonian quotients

Projective toric varieties are quotients of $\mathbb{C}^n$ by linear torus actions. Their mirrors can be described in those terms.

Example (Givental-Hori-Vafa mirror)

The best-known case is $\mathbb{P}^{n-1} = \mathbb{C}^n//\mathrm{U}(1)$, with mirror

$$(\mathbb{C}^*)^{n-1} = \left\{(z_1, \ldots, z_n) \mid z_1 z_2 \cdots z_n = q\right\}, \Psi = z_1 + \cdots + z_n$$

For $Y = \mathbb{C}^n$, with standard $(\mathbb{C}^*)^n$ action, declare the mirror to be

$$Y^\vee = (\mathbb{C}^*)^n, \quad \Psi = z_1 + \cdots + z_n.$$

For $X = \mathbb{C}^n//K$ with $K_\mathbb{C} \subset (\mathbb{C}^*)^n$, $X^\vee_\mathbf{q}$ is the fiber over $\mathbf{q} \in K^\vee_\mathbb{C}$ of the dual surjection $(\mathbb{C}^*)^n \twoheadrightarrow K^\vee_\mathbb{C}$, and the super-potential is the restricted Ψ.

($\mathbf{q}$ tracks degrees of holomorphic curves.)