Topological Gauge Theory
in low dimensions

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The main characters

This talk surveys, under the gauge theory umbrella, a number of intriguing topics in equivariant topology that arose over the past decades.

Necessarily, that leads to a retro-flavored lecture, but we will move up to recent results on *Coulomb branches* in the gauged linear Sigma-model and ongoing work on categorical representation theory. We will meet:

1. The Verlinde ring of a compact group, a 2D TQFT
2. Twisted equivariant K-theory, ${}^{\tau}K_G(G)$, $\tau \in H^4(BG;\mathbb{Z})$
3. Deformations of these structures: Higgs bundles, the gauged linear Σ-model;
4. Interpretation: Brauer group of (equivariant) K-theory
5. "Categorification" of ${}^{\tau}K_G(G)$: Equivariant matrix factorizations
6. The (conjectural) KRS 2-category of hyper-kaehler manifolds
7. Coulomb branches of 3D and 4D gauge theory

A compact group G and a class $\tau \in H^4(BG;\mathbb{Z})$ define a 2D *conformal field theory* with Hilbert space of states $\bigoplus H \otimes H^*$, over τ'-projective positive energy irrepresentations H of the free loop group LG.

On a conformal surface, the CFT is assembled from its chiral/anti-chiral components using a famous 3D topological field theory (Chern-Simons). Reduced over a circle, the latter gives the 2D *Verlinde theory* V.

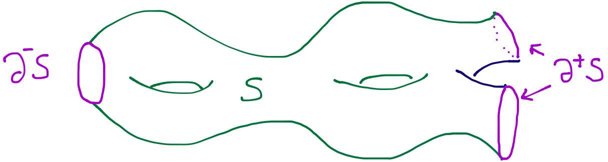
To S^1, V associates the Verlinde ring $R(G,\tau)$ of PERs with their *fusion product*, defined by merging a pair of S^1s to a single one via a conformal pair of pants. When $\pi_0 G = \pi_1 G = 1$, $R(G,\tau)$ is a quotient of $R(G)$, the representation ring, by an ideal which can be explicitly described.

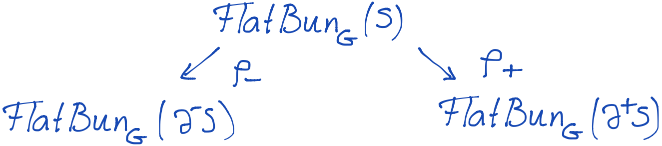
A theorem in topology (Freed, Hopkins, -) identifies $R(G,\tau)$ with the ring ${}^{\tau}K_G^{top}(G)$ with its Pontryagin product. Viewing $G \ltimes G$ is the stack of flat connections on S^1, the Pontryagin product is induced by the pair of pants.

A similar picture for a general surface defines operations on ${}^{\tau}K_G^{top}(G)$, defining a TQFT over the K-theory spectrum (enhanced from $\mathbb{C}$).

However, this enhanced theory is not *fully extended* (to points).

TQFT structure from topology





Correspondences coming from $\partial^{\pm}$-restriction of flat bundles define maps

$$(\rho_+)_! \circ \rho_-^* : {}^{\tau}K_G(G) \to {}^{\tau}K_G(G) \otimes {}^{\tau}K_G(G)$$

which (together with a certain trace) assemble to a K-linear TQFT.

Verlinde deformations

The theory V associates to a closed surface S the (dimension of the) space of Θ-functions for G at level τ. This can be enhanced to the Θ-sections over the moduli of Higgs bundles. To keep this finite, we filter by the degree (at infinity) and use the Poincaré series: hence the formal nature of this deformation in a parameter t. The theory returns to V at $t = 0$.

A closely relative is the *gauged linear Σ-model* associated to a unitary (complex and real-symplectic) representation E. Here, the space for S is the index of the Θ-line bundle over the holomorphic mapping space of S to the quotient stack $G \ltimes E$, instead of just the space of G-bundles. The mapping space consists of pairs: one G-bundle and one section of the associated E-bundle (plus an obstruction bundle from H^1):

$$\Gamma\left(Bun_G(S); \Theta^\tau \otimes \mathrm{Sym}\ \mathrm{Index}_S(E)\right).$$

Tracking symmetric degrees by t leads to the deformation of $V, R(G, \tau)$.

Twistings, higher twistings and deformations of ${}^\tau K_G^{top}(G)$

There are CFT deformations going with this, but the story can be understood purely topologically.

The datum $\tau \in H^4(BG)$ (whose transgression to $H^3_G(G)$ is the K-theory twist) is an element in the *Brauer group* of K-theory, $H^2(BG; GL_1(K))$.

Indeed, the group of lines $Pic \cong K(\mathbb{Z};2)$ is a subgroup of GL_1 of K-theory. But that has another, larger part $BSU_\otimes$ (a spectrum equivalent as a space to $BSU_\oplus$). The deformations we discussed come from here.

Specifically, there is a stable exponential map $BU_\oplus \to BU[\![t]\!]_\otimes$, the total symmetric power. Its 2-delooping, applied to $E \in BU^G_\oplus$, t-deforms τ.

A deformed version of the FHT theorem identifies the deformed ${}^\tau K_G^{top}(G)$ with the deformed Verlinde TQFT [-, Woodward].
This was proved by abelianization (reduction to the maximal torus and Weyl group and direct calculation, bypassing the formulation in terms of ${}^\tau K$, which is not written down.)

Remark

The Brauer group perspective also suggests a fix for the problem that ${}^{\tau}K_G^{top}(G)$-theory is not fully extended.
Namely, equivariant Brauer classes should give twisted versions of G-equivariant K-modules; or a G-equivariant Azumaya algebra over K, whose modules we then seek. This is what Brauer classes do over a ringed space. The point should correspond to this category of K-modules.
Unfortunately this has not been carried out to date. We seem to not find enough G-representations on K-modules.

Categorifying ${}^{\tau}K_G^{top}(G)$ over $\mathbb{C}$

The complexified ring of ${}^{\tau}K_G^{top}(G)$ is the Jacobian ring of the function

$$g \mapsto \Psi(g) := \frac{1}{2}\log^2 g + \mathrm{Tr}_E\left(\mathrm{Li}_2(tg^{-1})\right)$$

on regular conjugacy classes g, with the proviso that "critical point of Ψ" means "$d\Psi$ is a weight". We square in the quadratic form defined by τ. The formula generalized Witten's for topological Yang-Mills theory (which appears as $\tau \to \infty$).

Jacobian rings are Hochschild cohomologies of *Matrix factorization* categories associated to a superpotential, of brany fame in 2D mirror symmetry, where they define mirror models of Gromov-Witten theory. So this is a "mirror" of 2D gauge theory.

Remark

Matrix factorizations are 2-periodic curved complexes $E^0 \rightleftarrows E^1$, with the differential squaring to Ψ.

MFs appear in a different form in a categorification of the FHT theorem.

An invertible function (like e^{Ψ}) is an automorphism of the identity functor on category of sheaves on a space. That is the same as a $B\mathbb{Z}$-action, a rigid form of a circle action. The quotient stack $G \ltimes G$, a rigid version of the loop space LBG, has a natural $B\mathbb{Z}$=action — even after τ-twisting.

Theorem (Freed,-)

The G-equivariant, τ-twisted MF category over G with superpotential Ψ defined by the $B\mathbb{Z}$ loop-rotation action on $G \ltimes G$ is equivalent to the category of τ'-projective PERs of LG.

An indication of why comes by interpreting $G \ltimes G$ as flat connections, $LG \ltimes \Omega^1(S^1; \mathfrak{g})$. A formal Koszul duality argument converts this into the crossed product algebra $LG \ltimes \mathrm{Cliff}(L\mathfrak{g})$, with Dirac operator as (curved) differential. A basis of objects in the category consists of $H \otimes S^{\pm}$, where H runs over appropriate LG-representations and $S^{\pm}$ are the LG-spinors. In finite dimensions (compact groups), one proves the theorem this way. For loop groups, this is heuristic. (One reduces to finite dimension.)

Coulomb branches for 3/4D pure gauge theory

The following spaces, introduced by Bezrukavnikov-Finkelberg-Mirkovic, were key to reconciling the two MF pictures in gauge theory.

$$C_3(G) := \mathrm{Spec} H_*^{G^\vee}(\Omega G^\vee); \quad C_4(G) := \mathrm{Spec} K_*^{G^\vee}(\Omega G^\vee)$$

Theorem (BFM)

1. *Spec* $H_*^G(\Omega G)$ *is an affine resolution of singularities of* $(T^* T_{\mathbb{C}}^\vee)/W$.
2. *Spec* $H(\Omega G) \subset$ *Spec* $H_*^G(\Omega G)$ *is the fiber over* $Z(G^\vee) \subset (T^* T_{\mathbb{C}}^\vee)/W$.
3. *Spec* $H_*^G(\Omega G)$ *is algebraic symplectic, and Spec* $H_*(\Omega G)$ *Lagrangian.*
4. *Spec* $K_*^G(\Omega G)$ *is an affine resolution of singularities of* $(T_{\mathbb{C}} \times T_{\mathbb{C}}^\vee)/W$.
5. *Spec* $K(\Omega G) \subset$ *Spec* $K_*^G(\Omega G)$ *is the fiber over* $Z(G^\vee) \subset (T_{\mathbb{C}} \times T_{\mathbb{C}}^\vee)/W$.
6. *Spec* $K_*^G(\Omega G)$ *is algebraic symplectic, and Spec* $K_*(\Omega G)$ *Lagrangian.*
7. $C_3(G), C_4(G)$ *are the phase spaces of the Toda completely integrable systems for* G*, under projection to* $\mathfrak{t}_{\mathbb{C}}/W$*, resp.* $T_{\mathbb{C}}/W$.

Coulomb branches as classifying spaces for gauge theory I

These spaces carry a kind of *character theory* for 2-dimensional gauge theories, serving as *classifying spaces* for the latter. In examples coming from gauged Gromov-Witten theory, the character calculus permits the recovery of TQFT information (Seidel operators, J-function...).

More generally, C_3 carries the characters of topological G-actions on categories, which generate 2-dimensional gauge TQFTs over $\mathbb{C}$. C_4 should do that for K-linear categories (gauge TQFTs over K), or for LG-actions on linear categories.

No account of the K-theory version has been worked out beyond examples.

A precise statement is a (partially verified) equivalence between two 3D TQFTs: pure G-gauge theory PG versus Rozansky-Witten theory of C_3. The full statement for C_4 is less clear: LG-gauge theories are problematic in physics, while understanding K-linear gauge theory seems still a bit out of reach (to the speaker).

Coulomb branches as classifying spaces for gauge theory II

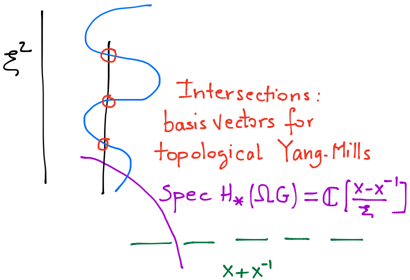
Nevertheless, precise consequences of this may be stated (and proved in many examples). Namely, gauged $2D$ theories give rise to a sheaf of categories with Lagrangian support on C_3 (C_4, in the K-linear case).

A construction has been proposed [KRS] of a 2-category of sheaves of categories on a hyper-Kaehler manifold, related to Rozansky-Witten TQFT. A precise claim would be that this KRS 2-category is equivalent to the 2-category of $\mathbb{C}$-linear categories with G-action.

In the present case, we view 2D gauge TQFTs as boundary conditions for PG. The equivalence with RW theory gives the correspondence between the two models of boundary conditions.

While the full program is not developed, portions can be established for $C_{3,4}$ using their integrable system structure ($\rightsquigarrow$ preferred co-ordinates). This allows one to show one direction of the correspondence.

Pictures instead of thousands of words: $G = \mathrm{SU}(2)$



C_3 with the unit, regular and Verlinde ($d\Psi$) Lagrangians. x, ξ are the coordinates on $T^\vee, \mathfrak{t}$.

The Toda projection to the axis ξ^2 of adj. orbits gives an integrable system, which is also an abelian group scheme over the base $\mathfrak{t}/W$. The unit section is in black.

C_4 is vertically periodicized; as is, it's good near $1 \in T^\vee$ at large level.

General 2D gauge TQFT are generated by categories with G-action and have a "character" in $C_{3,4}$. The group structure along the Toda fibers corresponds to the tensor product of TQFTs.

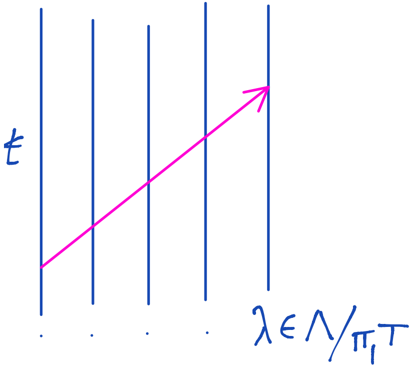
Verlinde rings in the Coulomb branches

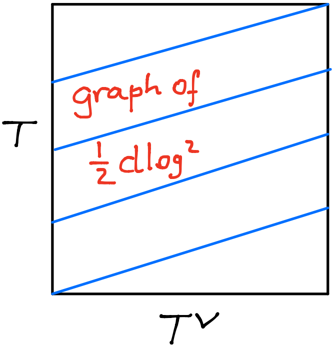
The Verlinde rings we discussed appear as Jacobian rings within $C_4(G)$; specifically, intersections of two Lagrangian subspaces, the "trivial representation" and the "Verlinde Lagrangians" $\exp(d\Psi)$.
(They acquire a Frobenius trace from the natural volume form.)

Note that $\exp(d\Psi)$ is the sum of a regular part from $\log(g)^2$, representing an isogeny $T \to T^\vee$ with kernel the undeformed Verlinde ideal, and a singular part from Li_2. The Li_2 Lagrangian is the *B-model mirror of E* as a symplectic manifold with G-action.

So far, this is only a restatement of the deformed Verlinde ring calculation: "$d\Psi$ is a weight" was imposed by periodicising $T^*T_\mathbb{C}$ to $T_\mathbb{C} \times T_\mathbb{C}^\vee$.

More is true, however: the $G \ltimes G$ twisted MF computation can also be mapped to C_4, as illustrated next for $U(1)$ (no twisting for simplicity)





Spec$(T \ltimes_\tau \mathbb{C}[T]) \cong (\mathfrak{t} \times \Lambda)/\pi_1$, differs from Spec$(T \ltimes \mathbb{C}[T]) = T \times \Lambda$ by coupling the π_1 action on $\mathfrak{t}$ to Λ. $\Psi = \frac{1}{2}\xi^2 + \lambda(\xi)$, with one critical point on each sheet.

$C_4(T) = T_{\mathbb{C}} \times T_{\mathbb{C}}^\vee$ with the graph of the Verlinde Lagrangian $\frac{1}{2} d \log^2$. Its intercepts with $T \times \{1\}$ are in bijection with the Verlinde points $\lambda/\pi_1 T$.

I have not discussed some current topics:

1. Coulomb branches for "3D gauge theory with matter": these are algebraic symplectic spaces, closely related to the $C_{3,4}(G)$ and depending on the representation E.
   Braverman-Finkelberg-Nakajima gave a precise definition. Can be much simplified with our ingredients when the "matter" is $E \oplus E^*$, but not for more general representations.
   That is an open problem (related to the existence of topological boundary conditions for gauge theory with matter)
2. Gauged linear Σ-model as a TQFT over varying surfaces.
   Partition function for surfaces $\rightsquigarrow$ a K-class over the moduli of curves.
   Several approaches exist, none yet carried out (let alone compared): the semi-simple TQFT classification theorem; Kontsevich, Saito et al theorem about primitive forms, Frobenius manifolds and TQFTs; and the abelianization method of (-, Woodward).
   The latter is in progress (for smooth curves); the answer involves Deligne-Mumford symmetric powers.

3. Coulomb branch C_4 as a space over K-theory.
   This would improve Verlinde theory to K-modules, from $\mathbb{C}$-modules.
   The K-linear theory lies in between 3D Chern-Simons theory and its $2D$ Verlinde reduction. Thus, for a surface, CS gives a vector space, ${}^{\tau}K_G(G)$ a point in K, and V a number. This is a weak form of the problem of assigning a reasonable topological object to CS(point).

4. Finally, I used several variants of gauge theory, whose their inter-relation is not so clear. (To me)
   CS theory and the pure 3D gauge theory PG are quite different.
   In fact, CS has very few (if any) topological boundary conditions, but a remarkable conformal boundary condition; whereas pure topological gauge theory PG has lots of topological ones. (Compact symplectic G-manifolds, and some non-compact ones, like E.)
   Often, these are disjoint phenomena ("gapped vs. ungapped"), yet PG can compute many Chern-Simons related quantities.