Representations of SL$(2,\mathbb{R})$ via brane quantization, 1

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Here we follow the discussion in Gukov and Witten [1]. We're going to use the A-model machinery we developed previously in an example, where we can get the representation theory of SU(2) and SL$(2,\mathbb{R})$ from quantization of our manifold. We'll start with a review of the A-model, and some remarks about its features that will be useful later, then introduce the example and choose different branes and parameters, which will give us the representation theory we want.

1 Review of the A-model

The A-model [2] is obtained by a twist of the $\mathcal{N}=(2,2)$ $(1+1)$-d σ-model. Here we will just consider the usual target for the A-model, i.e. some Kähler manifold Y. The σ-model has bosonic and fermionic fields: the bosonic fields are given by maps $\phi : \Sigma \to Y$, where Σ is the worldsheet, and there are four fermionic fields $\psi_\pm, \bar{\psi}_\pm$, and there are four supercharges $Q_\pm, \bar{Q}_\pm$. There are also two U(1) symmetries of the action, and we denote these by $\mathrm{U}(1)_V, \mathrm{U}(1)_A$

The sigma model depends on the Riemannian metric on Y, but turns out in this setting we can twist the theory so that it becomes independent of that metric. There are two inequivalent twists, the A-twist and the B-twist which we get by using the $\mathrm{U}(1)_V$ or the $\mathrm{U}(1)_A$ symmetry. In our case we're mostly concerned about the A-twist, but we will see that in the case where Y has a hyperkähler structure, it becomes useful to consider both models, since we can regard the A-twist in symplectic structure ω_J as a B-twist in complex structure K.

Most important to us is to describe the boundary conditions that we will allow for the A-model. Let's take Y to have a holomorphic symplectic structure $\Omega = \omega_J + i\omega_K$ and a complex structure I. We first pick a B-field, that is a class is $H^2(Y,\mathbb{C})$. We will need two kinds of boundary conditions:

- Lagrangian branes $\mathcal{B}'$, supported on Lagrangian submanifold of middle dimension (for ω_K). These carry a flat unitary Chan-Paton bundle $\mathcal{L}'$
- The coisotropic brane $\mathcal{B}_{cc}$, supported on all of Y and with a unitary Chan-Paton bundle $\mathcal{L}$ of curvature F satisfying $(\omega_K^{-1}(F+B))^2 = -1$, which in the hyperkähler case satisfies $F = \omega_J$

2 Spaces of strings

Given any A-branes $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ we have:

- A space of strings $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)$. This is given by quantizing the theory on a strip.

- A pairing $\text{Hom}(\mathcal{B}_1, \mathcal{B}_2) \otimes \text{Hom}(\mathcal{B}_2, \mathcal{B}_1) \to \mathbb{C}$, given by the twice punctured disk
- A composition law $\text{Hom}(\mathcal{B}_1, \mathcal{B}_2) \otimes \text{Hom}(\mathcal{B}_2, \mathcal{B}_3) \to \text{Hom}(\mathcal{B}_1, \mathcal{B}_3)$, given by the disk with three punctures
- An isomorphism between $\text{Hom}(\mathcal{B}, \mathcal{B})$ and the space of local operators that can be inserted in the brane $\mathcal{B}$

For details look at Kapustin-Witten section 11 [3].

Given the coisotropic brane $\mathcal{B}_{cc}$ and some Lagrangian brane $\mathcal{B}'$, we will look at the following spaces of strings:

- $\text{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc})$, which (in degree zero) is given by holomorphic functions on Y. This follows from the analysis of which local operators can be inserted on the coisotropic brane while preserving Q_A supersymmetry. In general, this is a function on Y, and on such functions the susy charge Q_A is given by $\bar{\partial}$ in complex structure I
- $\text{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$, which we want to be the space of quantum states. Calculating this space should be the same as doing geometric quantization with prequantum line bundle $\mathcal{N} = \mathcal{L} \otimes \mathcal{L}'^{-1}$. This is hard to calclate in general, but in the hyperkähler case this space admits a more explicit description, and as a vector space is given by $H^*(M, \sqrt{K} \otimes \mathcal{N})$. The Hilbert space structure doesn't come naturally from this and we need extra ingredients to describe it

3 The algebra of observables

Consider the following complexification of the 2-sphere, isomorphic to T^*S^2

$$Y = \{x^2 + y^2 + z^2 = \mu^2/4\} \subset \mathbb{C}^3$$

where μ is some complex constant. This is a hyperkähler manifold with the Eguchi-Hansen metric, and we can give it a holomorphic symplectic form $\Omega = \omega_J + i\omega_K = \frac{dy \wedge dz}{x}$. The manifold Y has an action of $\text{SO}(3, \mathbb{C})$ and Ω is invariant under this action. Note also that if we identify $\mathfrak{SO}(3, \mathbb{C}) \simeq \mathbb{C}^3$ and then Y is a regular coadjoint orbit.

Now let's look at the space of parameters for the A-model with this target. The Chan-Paton bundle $\mathcal{L}$ is determined by three periods around the cycle represented by the real sphere (the only nontrivial 2-cycle)

$$\alpha = \int_{S^2} \omega_I/2\pi, \beta = \int_{S^2} \omega_J/2\pi, \gamma = \int_{S^2} \omega_K/2\pi,$$

These are related to the parameter $\mu = \pm(\beta + i\gamma)$. The only other thing we have to determine is the class of the B-field, which is determined by $\eta = \int_{S^2} B/2\pi$. Not every choice leads to an A-brane: we need to give a Chan-Paton line bundle $\mathcal{L}$ with curvature F such that $\omega_K^{-1}(F + B)$ squares to -1, which in the HK case implies $F + B = \omega_J$. The isomorphism class of $\mathcal{L}$ is determined by an integer $n = \int_{S^2} F/2\pi = \int_{S^2} c_1(\mathcal{L})$, so we need to impose a the constraint $\beta - \eta = n \in \mathbb{Z}$

The choice of $(\alpha, \beta, \gamma, \eta)$ up to orientation reversal determines the coisotropic brane, which means it also determines our algebra of observables. This is an important point: our algebra of observables (and therefore in our case the value of the Casimir element) only depend on our choice of coisotropic brane, since it is given by $\text{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc})$

Classically, this ring is given by holomorphic functions on the support of the coisotropic brane Y in complex structure I. When we quantize we will get:

- A space of states $\mathcal{H}$ given by holomorphic functions on Y
- An algebra of operators $\mathcal{A}$ acting on $\mathcal{H}$. These operators are generated by J_x, J_y, J_z, which correspond to the observables x, y, z

In the quantized theory the algebra structure is different. It's a general result (Kapustin-Witten) that the product gets deformed by the Poisson bracket corresponding to the holomorphic symplectic form Ω. So $[J_x, J_y] = \{x, y\} = z$. The classical algebra also has another relation $x^2 + y^2 + z^2 = \mu^2/4$. This also corresponds to a relation between operators, which turns out is $J_x^2 + J_y^2 + J_z^2 = (\mu^2 - 1)/4$. That is, we get $\mathcal{A} = \mathcal{U}/I$ i.e. the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2, \mathbb{C}))$, mod the ideal I generated by the relation $J^2 = (\mu^2 - 1)/4$, where $J^2 = J_x^2 + J_y^2 + J_z^2$ is the Casimir.

So, for whatever other brane $\mathcal{B}$ we pick, we'll have an $\mathcal{A}$-module $\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B})$. Depending on which A-brane $\mathcal{B}$ that we pick, we're gonna get different $\mathcal{A}$-modules but the value of the Casimir will be the same. For example, if we pick the same brane $\mathcal{B}_{cc}$ we get again $\mathcal{A}$ with the algebra action on itself, i.e. an infinite dimensional $\mathfrak{sl}(2, \mathbb{C})$ representation. This doesn't necessarily mean that we'll get a group representation, since we're dealing with non-compact groups. However, note now that in the case where we pick $\mathcal{B}_{cc}$ itself, we do have an honest $G = \mathrm{SO}(3, \mathbb{C})$ representation: Y is G-invariant, so G acts on the functions generated x, y, z in the fundamental representation. This action extends to the quantized algebra, and G acts on $\mathcal{A}$. Differentiating this action, we get a $\mathfrak{g}$ action on $\mathcal{A}$ and we can check this action agrees with the action on $\mathcal{A}$ on itself.

So, if we can find another brane $\mathcal{B}'$ that is also G-invariant, we hope that the action of $\mathfrak{g}$ on $\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$ will exponentiate to an honest G-action. We don't need this, though: if we find some brane invariant under a subgroup H of G, we can at first restrict the action to $\mathfrak{h} \subset \mathfrak{g}$ and then we hope that this will exponentiate to an action of H. In the next example, we're gonna look at the cases where $H = \mathrm{SU}(2), \mathrm{SL}(2, \mathbb{C})$, and for that we will pick appropriate branes.

4 Hermitian structure/Unitarity

Now we want a way to assign a Hermitian structure to $\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}')$, and to check when the representation of H is unitary. First let's remember we already have a pairing between $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)$ and $\mathrm{Hom}(\mathcal{B}_2, \mathcal{B}_1)$, which is non-degenerate. So using this we can make a Hermitian product if we have a complex antilinear map $\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}') \to \mathrm{Hom}(\mathcal{B}', \mathcal{B}_{cc})$ Turns out it is possible to construct such a map if:

- We have an involution τ of Y that reverses the symplectic form $\tau^*\omega_K = -\omega_K$
- $\mathcal{B}_{cc}$ has to be τ-invariant. It's supported on all of Y, so we just need to check if its Chan-Paton bundle is τ-invariant, which happens if $\tau^*\omega_J = \omega_J$, i.e. $\tau^*\Omega = \bar{\Omega}$
- $\mathcal{B}'$ has to be τ-invariant. The easy way to do this is if its support $M \subset Y$ is fixed pointwise by τ. In that case, we get a positive-definite Hermitian product, and he representation we get is unitary. Another way to do this is if M is just τ-invariant, but then we need to pick an action on the Chan-Paton bundle, which involves a choice. In these cases, the representations can be unitary or not.

5 SU(2) representations

We'll work out an easy case first, then later get our hands dirty with picking other lagrangians. Pick $\mu^2 > 0$ real (which corresponds to $\gamma = 0, \beta \neq 0$) and no B-field, so $\eta = 0$ In coordinates (x, y, z), consider the involution τ of Y given by complex conjugation. The fixed point set of this is $M = S^2$ which represents the nontrivial 2-cycle of Y. This is Lagrangian for the symplectic structure ω_K, so we can put a $\mathcal{B}'$ there. There's also have an integrality restriction on μ, so that the coisotropic brane makes sense:

$$\mu = \beta = n = \int_M c_1(\mathcal{L})$$

This is invariant under the action of $\mathrm{SO}(3) \subset \mathrm{SO}(3, \mathbb{C})$, so we'd expect to get representations of SO(3). As is usual in QM, since we only care about projective representations, we get a little more: reps of the universal cover SU(2)

By our previous argument using the HK structure, this is the same as picking a prequantum line bundle $\mathcal{N} = \mathcal{L} \otimes \mathcal{L}'^{-1} = \mathcal{L}$ (notice that the bundle $\mathcal{L}'$ is flat and thus trivial on S^1). So we have

$$\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}') = H^*(M, \sqrt{K} \otimes \mathcal{L}) = H^*(\mathbb{P}^1, \mathcal{O}(n-1))$$

This sheaf has no higher cohomology so the total dimension is $\dim H^0(\mathbb{P}^1, \mathcal{O}(n-1)) = \binom{n}{1} = n$. We can easily guess which representation we have: it has highest weight $(n-1)/2$ and as we expected the Casimir acts as $(n^2 - 1)/4$. These exhaust already all the reps of SU(2).

References

[1] S. Gukov and E. Witten, *Branes And Quantization*, arXiv:hep-th/0809.0305v2

[2] K. Hori et al, *Mirror Symmetry*, AMS

[3] A. Kapustin and E. Witten, *Electric-Magnetic Duality and the Langlands Program*, arXiv:hep-th/0604151v3