1 Supersymmetric Quantum Mechanics

We follow section 10.5 of the Clay Math Mirror Symmetry book. Another reference is Ed Witten's Supersymmetry and Morse Theory

This classical aspects of this story are really dodgy because we have fermions, so we need odd complex numbers (which anticommute) and things like that, and quantization is really strange, so if you wish you can just skip to the quantum parts.

1.0.1 classical

We begin by considering a particle propagating in a compact Riemannian manifold (M, g). We denote the position of the particle by a function $\phi : \mathbb{R} \to M$. The Lagrangian (which in this case is just the kinetic energy) is given by

$$L = g(\dot{\phi}, \dot{\phi}),$$

where dot denotes time derivative. We will add to our configuration a complex "fermi" field $\psi \in \Gamma\phi^*TM \otimes \mathbb{C}^{odd}$ and modify the Lagrangian to

$$L = g(\dot{\phi}, \dot{\phi}) + ig(\bar{\psi}, D_t\psi) - ig(D_t\bar{\psi}, \psi) - R(\psi, \bar{\psi}, \psi, \bar{\psi}),$$

where D_t is the one-dimensional Dirac operator induced by the metric on M

$$D_t\psi = \partial_t\psi + \Gamma(\partial_t\phi, \psi),$$

and R is the Riemann curvature tensor. This is done so that the action $S = \int Ldt$ is invariant under the infinitesimal transformations generated by

$$\delta\phi = \epsilon\bar{\psi} - \bar{\epsilon}\psi$$

$$\delta\psi = \epsilon(i\dot{\phi} - \Gamma(\bar{\psi}, \psi)).$$

The first of these should be a tangent vector to the space of configurations of ϕ, in other words a section of ϕ^*TM. The parameter ϵ is odd but is a scalar, so the two terms are both sections of $\phi^*TM \otimes \mathbb{C}$. The rules of $\mathbb{C}^{odd}$ say that $\bar{ab} = \bar{b}\bar{a} = -\bar{a}\bar{b}$, so the combination in $\delta\phi$ is real, as required. The second should be a section of $\phi^*TM \otimes \mathbb{C}^{odd}$, and indeed both $\dot{\phi} \in \Gamma, \Gamma(\bar{\psi}, \psi) \in \Gamma\phi^*TM$, while ϵ is odd.

This is called a supersymmetry because the parameter ϵ is odd. Noether's theorem works for such things in the ordinary way, and we obtain a corresponding "fermionic" conserved charge

$$Q = ig(\bar{\psi}, \dot{\phi}),$$

which is an observable–a function on our space of configurations–but it takes odd value.

This theory also has a $U(1)$ symmetry

$$\psi \mapsto e^{-i\theta}\psi.$$

This symmetry is called fermion number, which will make more sense when we quantize.

The conjugate momenta to ϕ and ψ, respectively, are

$$p = g(\dot{\phi}, -)$$

$$\pi = ig(\bar{\psi}, -).$$

An observable measured at a time t_0 is a complex-valued (perhaps odd) smooth function of $\phi(t_0), p(t_0), \psi(t_0), \pi(t_0)$ (this is a locality constraint: observables are not arbitrary functionals). Note because $\bar{\psi}$ is basically the conjugate variable to ψ it is natural to treat them as independent quantities when writing down observables. When we quantize it is customary to assign an operator to $\bar{\psi}$ rather than π, which will introduce some asymmetry in the construction.

For example, the supercharge

$$Q(t_0) = i\bar{\psi}(t_0)p(t_0)$$

evaluted at a time t_0 is an odd complex number.

There is a (super) Poisson structure on such things determined in a region with coordinates by

$$\{\phi^i, p_j\} = \delta^i_j$$

$$\{\psi^i, \pi_j\} = \delta^i_j.$$

All others are zero.

1.0.2 classical to quantum

Now let's quantize (there are many choices! this is just the usual way! take this data as the fundamental thing!). We must invent operators on some Hilbert space for each of our observables such that the commutator is given by the Poisson bracket. The Hilbert space is $\mathcal{H} = \Omega(M, \mathbb{C})$ with Hermitian form given by integrating against the volume form of M (which is why M is compact).

- To an observable only depending on the value of ϕ, ie. a real-valued function on M, the operator is multiplication by that function.
- To an observable only depending on the value of p, ie. a vector field on M, the operator is covariant derivative along that vector field.
- To an observable only depending (linearly) on $\bar{\psi}$, ie. a 1-form, the operator is wedge with that 1-form.
- To an observable only depending (linearly) on ψ, ie. a 1-form, the operator is contract with the metric dual of that 1-form.

Observables that are functions of variables that are not supposed to commute require more choices, but for now we are not interested in such observables (and indeed arbitrary observables are something that needn't be considered).

An example of something we are interested in is the supercharge

$$Q = i\bar{\psi}p.$$

According to the rules above this gets promoted to the exterior derivative d. Its complex conjugate gets promotes to its adjoint $d^\dagger$.

1.0.3 just quantum

So our system has Hilbert space $\mathcal{H} = \Omega(M, \mathbb{C})$ with inner product

$$\langle \alpha | \beta \rangle = \int_M \star\alpha \wedge \beta.$$

It has two super charges Q and $\bar{Q}$ which act as

$$Q|\alpha\rangle = |d\alpha\rangle$$

$$\bar{Q}|\alpha\rangle = |d^\dagger \alpha\rangle.$$

The Hamiltonian is given by

$$H = \{Q, \bar{Q}\} = dd^\dagger + d^\dagger d = \Delta.$$

We are interested in the space of ground states, ie. those annihilated by H. These are precisely the harmonic forms on M. A general fact about supersymmetric theories (which follows from the above relation and Hodge theory type arguments) is that the space of ground states maps isomorphically onto the Q-cohomology of states ($Q^2 = 0$ is also a general feature, though we will see it fail soon enough). For us, this is of course $H^*(M, \mathbb{C})$.

Now let's turn on a potential $h : M \to \mathbb{R}$, which we will take to be generic (more on what this means later). This modifies the supersymmetry transformations to

$$Q = e^{-h} d e^{h}.$$

The cohomology of this operator (the space of ground states) is isomorphic to the cohomology of d, so turning on h does not change the space of ground states. Let us try to reproduce this result semi-classically, which will use tools that will readily generalize to the 2d case.

If we rescale the potential $h \mapsto \lambda h$, $\lambda \gg 1$, then classical configurations will stay close to the critical points of h. Each critical point thus gives rise to a ground state in perturbation theory, where we expand the Hamiltonian order by order around each critical point. In detail, for a critical point x_0, we can consider the Hamiltonian acting on differential forms falling off exponentially fast around the critical point. For this subspace of the whole Hilbert space, we can write

$$H_0 = \sum p_i^2 + \lambda^2 c_i^2 (x^i)^2 + \lambda c_i [\bar{\psi}^i, \psi^i] + \mathcal{O}(x^3),$$

where x_i are coordinates around x_0 where the Hessian of h is diagonal with eigenvalues c_i. This Hamiltonian has a unique ground state given by the differential form

$$|\Phi_0\rangle = e^{-\lambda |c_i| (x^i)^2} \prod_{j : c_j < 0} dx^j.$$

Note the form degree equals the Morse index of x_0. This ground state can be refined to all orders in perturbation theory (order by order in x in H_0). We call the result a perturbative ground state.

Do all perturbative ground states define true ground states? The answer must be no, since turning on the potential should not affect the space of ground states. What happened? To start, we should attempt to compute

$$Q|\Phi_0\rangle = \sum_j \langle\Phi_j|Q|\Phi_0\rangle|\Phi_j\rangle + \mathcal{O}(1/\lambda).$$

On the right we have expanded this in a basis including all the perturbative ground states Φ_j. Other terms in this expansion involve states with positive energy in perturbation theory, hence are supressed by λ. We compute in the limit $\lambda \to \infty$, so we can disregard such terms.

Thus, we need to compute the correlation function

$$\langle\Phi_j|Q|\Phi_0\rangle = \frac{1}{h(x_0)-h(x_j)}\langle\Phi_j|[Q,h]|\Phi_0\rangle$$

$$= \lim_{T\to\infty}\frac{1}{h(x_0)-h(x_j)}\langle\Phi_j|e^{-TH}[Q,h]e^{-TH}|\Phi_0\rangle$$

$$= \lim_{T\to\infty}\frac{1}{h(x_0)-h(x_j)}\langle\text{any state localized at } x_j|e^{-TH}[Q,h]e^{-TH}|\text{any state localized at } x_0\rangle$$

$$= \frac{1}{h(x_0)-h(x_j)}\int_{\phi(-\infty)=x_j}^{\phi(\infty)=x_0} D\phi D\psi D\bar{\psi}\ \partial_i h(\phi(0))\bar{\psi}^i(0)\ e^{-S_E}.$$

In the first line we have made some innocuous subsistution. The second and third lines use the fact that e^{-TH} in the limit $T\to\infty$ is an operator that projects onto the space of perturbative ground states. We can re-interpret e^{-TH} as an imaginary-time propagator. Then the correlation function reads "start with a state localized at x_j, evolve for a long time, then apply the operator $[Q,h]$, then evolve for a long time, then take the inner product with a state localized at x_0". This is exactly what is computed by the Euclidean-time path integral in the last line.

There are some important features to note about this. First, S_E is invariant under the $U(1)$ symmetry, while the integrand transforms

$$\partial_i h(\phi(0))\bar{\psi}^i \mapsto e^{i\theta}\partial_i h(\phi(0))\bar{\psi}^i.$$

Thus, in order for the integral to be nonzero, the measure must transform. This is called an Adler-Bell-Jackiew (ABJ) anomaly. The anomaly is due to the charge of zero modes. The measure above splits into a piece integrating over modes with positive kinetic energy $g(\psi, D_-\bar{\psi})$ and modes in the kernel of D_-. This is the operator

$$D_t - \phi^* H_h,$$

where the second term is the pulled back Hessian of h. The positive energy part of the measure is taken to be invariant, while the zero mode measure transforms opposite to the fields they are composed of. In other words, for the measure to have $U(1)$ charge -1 to cancel the variation of the integrand, we must have

$$\dim \bar{\psi} \text{ zero modes } - \dim \psi \text{ zero modes } = \dim \ker D_- - \dim \ker D_-^\dagger = \mathrm{ind} D_- = 1.$$

One can show that for any map $\phi : R \to M$ with $\phi(-\infty) = x_j$, $\phi(\infty) = x_0$, that the index of D_-, a connection on $\phi^* TM$, is the difference in Morse indices between these critical points.

The point is anomaly matching gives selection rules for the path integral.

The other symmetry we have at our disposal is the supersymmetry transformations. The measure and action are invariant under supersymmetry transformation by either parameters ϵ and $\bar{\epsilon}$ independently, but the integrand is only invariant under the ϵ transformations (note that such a transformation does not integrate to a valid transformation of ϕ, but we can think of it in an analytic continuation of the path integral). **Only δ_ϵ-invariant configurations contribute to the path integral.** All others can be transformed to one where the fermions do not appear in the integrand. These are zero by the rules of Grassman integration, see Clay Math Mirror Symmetry chapter 9. This is called localization.

The configurations invariant under Q are those satisfying $\bar{\psi} = 0$, $i\dot{\phi} = \nabla h$. Note that these are also minimal energy configurations. Configurations preserving half the supersymmetries (in this case Q but not $\bar{Q}$) are called BPS. Consider a nearby configuration $\phi + \xi$, $\psi + \chi$. The quadratic approximation to the action is

$$S_E = \lambda(h(x_0) - h(x_j)) + \int g(D_-\xi, D_-\xi) + g(D_-\bar{\psi}, \psi) + \mathcal{O}(\xi^3).$$

The Guassian integral over the nonzero bosonic modes ξ introduces a factor of $1/\sqrt{\det D_-^\dagger D_-}$, while the Gaussian integral over the two nonzero fermionic modes $\bar{\psi}, \psi$ produces a factor of $\det D_-$. Thus, these integrals together produce a sign (which we must work a bit harder to determine). A second consequence of supersymmetry is that **this quadratic approximation is exact**.

The path integral is now reduced to an integral over zero modes. There is a $\bar{\psi}$ zero mode which we can solve for using the definition of D_- above:

$$\bar{\psi}^i = \dot{\phi}^i \bar{\psi}_0,$$

where $\bar{\psi}_0$ is an odd normalization constant, and a ξ zero mode which corresponds to reparametrizing $\phi(\tau) \mapsto \phi(\tau + \xi_0)$. Integrating over this mode is equivalent to integrating over where the operator $\partial_i h \bar{\psi}^i$ is inserted. The path integral is now

$$\frac{1}{h(x_0) - h(x_j)} \sum_{\text{gradient flows } \phi} \pm^{\phi} \int d\bar{\psi}_0 \int d\tau \bar{\psi}_0 \dot{\phi}^i \partial_i h(\phi(\tau)) e^{-\lambda(h(x_0)-h(x_j))}$$

$$= \sum_{\text{gradient flows } \phi} \pm^{\phi} e^{-\lambda(h(x_0)-h(x_j))}.$$

Thus we have

$$Q|\Phi_0\rangle = \sum_{\text{flows to critical points with one higher Morse index } x_j} \pm e^{-\lambda(h(x_0)-h(x_j))} |\Phi_j\rangle.$$

The exponential can be absorbed into the wavefunction Φ_j. With some more work, one can figure out how to assign the sign to each flow. Then one finds $Q^2 = 0$ and we get another complex whose cohomology is $H^*(M, \mathbb{C})$, the space of true ground states.

2 The A-model

We begin with the supersymmetric 2d sigma model with target a Kahler manifold M. Besides the bosonic field $\phi : \Sigma \to M$, there are four fermions $\psi_\pm, \bar{\psi}_\pm \in \Gamma\phi^*TM \otimes_{\mathbb{R}} S_\pm$ (as before, the field and its complex conjugate are considered separate degrees of freedom, though they end up being antiparticles). The Lagrangian is

$$L = g(d\phi, \bar{d\phi}) + ig(\bar{\psi}_-, (D_0+D_1)\psi_-) + ig(\bar{\psi}_+, (D_0-D_1)\psi_+) + \text{four fermion terms}$$

where g is the Hermitian pairing on TM.

This theory has four supercharges $Q_\pm, \bar{Q}_\pm$. The supersymmetry action is

$$\delta\phi = \epsilon_+\psi_- - \epsilon_-\psi_+$$

$$\delta\psi_+ = 2i\bar{\epsilon}_-\partial_+\phi + \epsilon_+\Gamma(\psi_+, \psi_-)$$
$$\delta\psi_- = -2i\bar{\epsilon}_+\partial_-\phi + \epsilon_-\Gamma(\psi_+, \psi_-).$$

The theory has two $U(1)$ symmetries

$$U(1)_V : \psi_\pm \mapsto e^{-i\theta}\psi_\pm$$

$$U(1)_A : \psi_\pm \mapsto e^{\mp i\theta}\psi_\pm.$$

The first is called the vector symmetry, the second is called the axial symmetry.

The parameters of supersymmetry transformations $\epsilon_\pm, \bar{\epsilon}_\pm$ in the sigma model are (complex) worldsheet spinors since we need to form pairings like

$$\epsilon_\pm\psi_\mp$$

to get something in $\Gamma\phi^*TM \otimes_\mathbb{R} \mathbb{C}$. This can be done if ϵ is a spinor using the $Spin$-invariant pairing $S_+ \otimes_\mathbb{C} S_- \to \mathbb{C}$. In order for this to be a symmetry of the action, these parameters must be covariantly constant. Covariantly constant spinors only exist on flat Riemann surfaces, but we would like a theory that has a supersymmetry on curved Riemann surfaces as well. One way to do this is to modify the worldsheet spin of ψ so that it is a worldsheet scalar so in the above variation ϵ can also be a worldsheet scalar (and covariantly constant scalars always exist).

Put another way, every field is a section of some bundle associated to the $SO(2)$ frame bundle on Σ by a $Spin(2)$ representation. Each field also carries a representation of the vector symmetry $U(1)_V$. $Spin(2) = U(1)$, so we can and will twist this representation by the $U(1)_V$ representation. For example, the field ψ_- has $Spin(2)$ charge -1 and $U(1)_V$ charge 1, so it will become twisted to a worldsheet scalar field $\chi \in \Gamma\phi^*TM^{1,0}$. Likewise, $\bar{\psi}_+$ becomes a scalar $\bar{\chi} \in \Gamma\phi^*TM^{0,1}$, the other two spinors become worldsheet 1-forms ρ_z, $\rho_{\bar{z}}$, and ϕ, which does not transform under $U(1)_V$, is unchanged.

We rewrite the Lagrangian

$$L = g(d\phi, \bar{d}\phi) - ig(\rho_z, \partial_{\bar{z}}\chi) + ig(\rho_{\bar{z}}, \partial_z\bar{\chi}) - \frac{1}{2}R(\rho_{\bar{z}}, \chi, \rho_z, \bar{\chi}).$$

This action is invariant under the two $U(1)$ symmetries as well as half the supersymmetry transformations (corresponding to the supercharges that are now scalars):

$$\delta\phi = \epsilon_+\chi$$
$$\delta\bar{\phi} = \bar{\epsilon}_-\bar{\chi}$$
$$\delta\phi_{\bar{z}} = 2i\bar{\epsilon}_-\partial_{\bar{z}}\phi + \epsilon_+\Gamma(\rho_{\bar{z}}, \chi)$$
$$\delta\phi_z = -2i\epsilon_+\partial_z\bar{\phi} + \bar{\epsilon}_-\Gamma(\rho_z, \bar{\chi})$$
$$\delta\chi = \delta\bar{\chi} = 0.$$

Let Q_A be the supercharge parametrizing the combined transformation $Q_A = \bar{Q}_+ + Q_-$.

The twisted theory agrees with the sigma model on flat Riemann surfaces, including $S^1 \times \mathbb{R}$, so it can be quantized using the same Hilbert space with the same algebra of point operators. We will restrict the twisted theory to the space of Q_A-closed states and only consider Q_A-closed operators acting on them. This theory is called the A-model.

Note that Q_A-exact operators only have zero matrix elements between Q_A-closed states, so what really acts in this theory is Q_A-cohomology of operators. Because we restricted to ground states, the Hamiltonian of the A-model is identically zero. In fact, the whole stress-energy tensor is Q_A-exact, so correlation functions in the A-model do not change if we vary the worldsheet metric! We call the A-model a TQFT for this reason. In fact, things we can measure about the A-model

Let us determine this in this case. We are interested only in point operators right now, and these are made from ϕ and χ. Any such operator locally looks like

$$\omega(\phi)_{i_1,\ldots,i_n,\bar{j}_1,\ldots,\bar{j}_m}\chi^{i_1}\ldots\chi^{i_n}\chi^{\bar{j}_1}\ldots\chi^{\bar{j}_m}$$

evaluated at a point x. This is metric dual to a covector of type (n, m) at $\phi(x)$. In fact, one can check that Q_A acts as the de Rham differential on these objects, so the Q_A-cohomology of point operators is $H^*(M, \mathbb{C})$.

As a vector space, this is in fact isomorphic to the space of ground states of either theory. This is because of the state-operator correspondence: given a point operator, we can insert the operator at the mouth end of an infinite cigar. The path integral then gives a value for every choice of boundary condition at infinity. We think of this is the wavefunction evaluated on a classical configuration. In other words, this produces a state at the lit end of the cigar. This map is an isomorphism for TQFTs in every case I've seen. It is often an isomorphism for CFTs as well, but not always. See Katz and Sharpe, D-branes, open-string vertex operators, and Ext groups.

However, given two point operators, we can fuse them together to get another point operator. This operation endows $H^*(M, \mathbb{C})$ with a ring structure. We will see that this is NOT the ring structure we normally have. In general, we can expand the fusion product as

$$\lim_{y\to x} \mathcal{O}_i(x)\mathcal{O}_j(y) = \sum_k \lim_{y,z\to x} \langle \mathcal{O}_i(x)\mathcal{O}_j(y)\mathcal{O}_k(z)\rangle \mathcal{O}_k(x) + [Q, \Lambda].$$

There are some reasons to be worried about how to take these limits. In more complicated situations, with more operators being fused, we have to keep track of the combinatorics of what order things can fuse in. This gives, for example, Kontsevich's recurrence for genus 0 Gromov-Witten invariants. For the expansion above, however, we will find only genus 0 contributions, and the conformal group acts 3-transitively on the sphere, so the insertion points don't matter.

Let us consider $M = \mathbb{P}^1$ with its standard Kahler structure with volume t. Let H be the form Poincaré dual to a point. We want to compute the fusion product $H * H$. This means computing the path integral

$$\int D\phi D\chi D\rho H(x)H(y)H(z)e^{-S}.$$

As in the quantum mechanical case, this path integral will have selection rules coming from the two $U(1)$ symmetries and will localize on Q_A-invariant configurations. First, the $U(1)_V$ symmetry is not anomalous since zero modes always come in oppositely charged pairs. The charge of an operator is its holomorphic form degree minus its antiholomorphic form degree. H is type $(1,1)$, so the integrand is invariant under $U(1)_V$, as required. The axial symmetry $U(1)_A$ however *is* anomalous. The path integral measure transforms by $e^{2ik\theta}$ where

$$k = \#\chi \text{ zero modes } - \#\rho \text{ zero modes},$$

where # means dimension of the space of zero modes. By the index theorem,

$$k = c_1 M \cdot \beta + \dim M(1-g),$$

where β is the class of the image of the worldsheet Σ under ϕ, and g is the genus of the worldsheet. In the current case, H has axial charge 2, so for the integral to be nonvanishing, we need $k = 3$. This means the integral is restricted to maps of genus 0.

The operators $H(x)$ are Dirac delta distributions which are zero unless x is sent to a marked point in the target $\mathbb{P}^1$ (changing the target point changes the operator by something Q_A-exact). Thus, as in the quantum mechanical case, the path integral localizes on supersymmetric genus 0 configurations with the three points x, y, z sent to the marked points in $\mathbb{P}^1$. The supersymmetric configurations are the ones with minimal energy. The energy is the area, so these are the holomorphic maps. Because $SL(2, \mathbb{C})$ acts 3-transitively on $\mathbb{P}^1$, there is exactly one degree 1 map with three marked points $\mathbb{P}^1 \to \mathbb{P}^1$. The action of this map is the volume of the target $\mathbb{P}^1$, so

$$\langle HHH \rangle = e^{-t}.$$

With some more work (because they are quite degerate) one can show the other structure constants in the ring of point operators are the same as the cup product in $H^*(\mathbb{P}^1)$. The only place where it differs is the above correlation function, which causes

$$H \cdot H = He^{-t}.$$

This is called the quantum cohomology ring of $\mathbb{P}^1$. It is a one-parameter deformation of the ordinary cohomology ring of $\mathbb{P}^1$. The parameter of the deformation is the size of the target space. This is a first clue that the A-model has something to do with quantization.

3 A-branes

Let us attempt to extend the A-model to a Riemann surface with boundary. We follow Kapustin-Orlov Remarks on A-branes, mirror symmetry, and the Fukaya category. Another good reference is chapter 19 of the Clay Math mirror symmetry book.

We will be looking for boundary conditions that preserve the A-model supersymmetry $Q_A = Q_- + \bar{Q}_+$. Let Y be a submanifold of M. A common boundary condition in all sorts of systems restricts $\phi : \partial\Sigma \to Y$. Y is called a D-brane. This is not a complete boundary condition yet. A useful way to phrase a boundary condition such as this one is to say we have an endomorphism

$$R : TM|_Y \to TM|_Y$$

and we impose

$$d\phi|_{\partial\Sigma} = R(d\phi|_{\partial\Sigma}),$$
$$\psi|_{\partial\Sigma} = R(\bar{\psi}|_{\partial\Sigma}).$$

Using the metric, we can decompose $TM|_Y = TY \oplus NY$. Then we can write

$$R = (-id_{NY}) \oplus (g - F)^{-1}(g + F),$$

where F is a 2-form (regarded as an antisymmetric endo of TY) which we will take to be the curvature of a line bundle on Y (this is far from the most general sort of boundary condition we can write down, but it will do for us).

Where did this curvature come from? Sneakily I've added a new term to the sigma model action,

$$2\pi i \int_\Sigma \phi^* B,$$

where B is a fixed class in $H^2(M, \mathbb{C})$. On a Riemann surface with boundary, the above term is not gauge invariant under $B \mapsto B + d\lambda$, so we need to add the term

$$2\pi i \int_{\partial\Sigma} \phi^* A,$$

where A is a connection on a line bundle living on Y (where the boundary maps to) whose curvature is $B|_Y$. Physically, we say that the end-point of the string is electrically charged as it moves along Y. The boundary condition above says that the end-point moves along Y and further does not deliver any momentum to the brane.

Let us write the Kahler (1,1) form on M as ω. One can write the supercharges

$$Q_\pm = ig(\psi, d\phi) \pm \omega(\psi, d\phi).$$

For the boundary condition to preserve some supersymmetry, R must map supercharges to supercharges. One can check

$$R^T g R = g,$$

ie. R is an orthogonal transformation, so the imaginary part of each Q is preserved. We want Q_A to be invariant under R, so we need

$$R^T \omega R = -\omega.$$

A submanifold Y with a line bundle whose curvature defines an R satisfying the above equation is called a rank one A-brane.

Let us investigate the A-brane condition in more detail. For the splitting $TM|_Y = TY \oplus NY$, write

$$\omega^{-1} = \begin{pmatrix} A & B \\ -B^T & C \end{pmatrix}$$

.

The A-brane condition becomes

$$A = 0$$

$$BF = 0$$

$$gCg = FCF.$$

The first condition means that Y is coisotropic: $\omega|_Y$ has constant rank and the dimension of its kernel is the codimension of Y. The second condition says that F restricted to this kernel is zero. The third condition says that $\omega^{-1}F$ induces a complex structure on the cokernel.

The basic A-branes are given by Lagrangian submanifolds Y, in which case $C = 0$ so we can take $F = 0$.

Now we follow Gukov-Witten Branes and Quantization. Let (M, ω) be a symplectic manifold. A complexification is a complex manifold Y with a holomorphic (2,0) form Ω and an antiholomorphic involution $\tau : Y \to Y$ such that

- $\tau^*\Omega = \bar{\Omega}$,
- M is a component of the fixed point set of τ,
- $\Omega|_M = \omega$,
- Re Ω is the curvature of a line bundle to which the action of τ extends.

We consider the A-model with target Y and symplectic form $\omega_Y = \text{Im}\ \Omega$. M is Lagrangian and therefore defines an A-brane Gukov-Witten call $\mathcal{B}'$. This theory also has a space-filling coisotropic A-brane with $F = \text{Re}\ \Omega$. One can check $(\text{Im}\ \Omega)^{-1}\text{Re}\ \Omega$ squares to -1. (This is a different complex structure, J. often the two complex structures assemble into a hyperKahler structure). Gukov-Witten call this brane $\mathcal{B}_{cc}$.

Let us compute the space of ground states of strings stretching from the coisotropic to itself,

$$\mathcal{A} = \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{cc}).$$

We imagine quantizing the A-model on a ribbon geometry with the $\mathcal{B}_{cc}$ boundary condition on the top and bottom edges. This can be done by shrinking the ribbon to a line, ie. we reduce to the quantum mechanical problem of a particle propagating on the coisotropic brane. Indeed, the string has some tension (the energy has a factor of the area of the world-sheet) so the classical ground states have zero length (the open strings look like particles). The action for the A-model on the disk looks like

$$\int \phi^*(\omega - iB) + \int \{Q_A, V\} = \int_{\partial\Sigma} A + \int \{Q_A, V\}.$$

Thus, up to Q_A-exact things, the physics really is that of (the holomorphic version of) a particle propagating with no Hamiltonian.

We can appeal to the ordinary quantum mechanical result that

$$\mathcal{A} = C^\infty(Y, \mathbb{C}).$$

However, because we are really considering strings, $\mathcal{A}$ has an algebra structure that ordinary particle state spaces don't possess. Note that this algebra is necessarily associative but might not be commutative. Put another way, the above is really the algebra of observables of the quantum mechanical system, entering here because of the state operator correspondence we have in the 2d theory. See Kapustin-Witten section 11.1.

This algebra (which is considered the quantum mechanical algebra of observables) acts naturally on

$$\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}').$$

Our involution induces a map $\tau : \mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}') \to \mathrm{Hom}(\mathcal{B}', \mathcal{B}_{cc})$ and therefore a pairing on these strings that lands in

$$\mathrm{Hom}(\mathcal{B}', \mathcal{B}').$$

In most cases, the above can be identified as the cohomology of M, so taking the degree 0 piece (so that we land in $\mathbb{C}$) defines a Hermitian form for which the action of $\mathcal{A}$ is unitary.

Thus, we have produced an algebra of observables and a Hilbert space on which that acts by a unitary representation.

4 comparison with geometric quantization

Suppose we have a hyperKahler structure on Y extending its holomorphic symplectic structure (Ω, I). Then both our coisotropic and our Lagrangian brane are branes of type (A, B, A), ie. they are A-branes for the A-models with symplectic structure ω_I, ω_K and B-branes for the complex structure J. As B-branes, $\mathcal{B}'$ is represented by the structure sheaf of M, and $\mathcal{B}_{cc}$ is represented by the line bundle $\mathcal{L}$ with curvature $\omega = \omega_J$. Then it is a standard result

$$\mathrm{Hom}_B(\mathcal{O}_M, \mathcal{L}) = \mathrm{Ext}(\mathcal{O}_M, \mathcal{L}) = H^*(M, \mathcal{L} \otimes \sqrt{K}).$$

This agrees with the A-model calculation *as a vector space*, but τ does not give the right sort of mapping for this space to get an inner product in the B-model. For the factor of $\sqrt{K}$ see Freed and Witten.

References

[1] Clay Math Mirror Symmetry

[2] Katz and Sharpe, D-branes, open-string vertex operators, and Ext groups

[3] Witten, Supersymmetry and Morse Theory

[4] Witten, Topological Sigma Models

[5] Kapustin and Orlov, Remarks on A-branes, mirror symmetry, and the Fukaya category

[6] Gukov and Witten, Branes and Quantization

[7] Kapustin and Witten, Geometric Langlands

[8] Freed and Witten, Anomalies in String Theories with D-branes