Old Midterm Solutions

Question 1
(a) no, check. (b) Yes, it's $\mathrm{Re}(\exp(2e^z))$.

Question 2
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Question 3
With $z = \exp(\pi i/n)$ the imaginary part of the left hand side is the sum of the sines, whereas the right becomes

$$\mathrm{Im}\frac{1+1}{1-\cos(\pi/n) - i\sin(\pi/n)} = \mathrm{Im}\frac{2(1-\cos(\pi/n)+i\sin(\pi/n))}{(1-\cos(\pi/n))^2+\sin^2(\pi/n)} = \frac{\sin(\pi/n)}{1-\cos(\pi/n)}$$

I think the sign is wrong in the posted version.

Question 4

Question 5
(a) $1/2$; (b) $1/\sqrt{2}$; (c) 1; all work using the ratio test.

Question 6
The nth derivative at $z=0$ is $n! \cdot a_n$, so all coefficients must vanish.

Question 1

$$z^2 = \frac{\sqrt{2}+1}{2} - \frac{\sqrt{2}-1}{2} + \mathrm{i} = 1+\mathrm{i}, z^4 = 2\mathrm{i},$$

Whence $z = 2^{1/4}(\cos\frac{\pi}{8} + \mathrm{i}\sin\frac{\pi}{8})$ and $z^7 = 2^{7/4}(\cos\frac{7\pi}{8} + \mathrm{i}\sin\frac{7\pi}{8})$

Question 2
We have $1+\omega+\omega^2+\cdots+\omega^{n-1} = \frac{1-\omega^n}{1-\omega} = 0$. Now, use De Moivre's formula for each power ω^k and take real parts.

Question 3
(a) $f = u+iv$ is holomorphic if and only if it is real-differentiable and satisfies the Cauchy Riemann equations $u_x = v_y, u_y = -v_x$. If so, $u_{xx} = (v_y)_x = (v_x)_y = -u_{yy}$ by the equality of the mixed partials, which follows from continuity of second derivatives.
(b) Note that $xy/(x^2+y^2)^2 = \mathrm{Im}(z^2/(z\bar{z})^2) = -\mathrm{Im}(\bar{z}^2/(z\bar{z})^2) = \mathrm{Im}(1/z^2)$, which is holomorphic away from $0 \in \mathbb{C}$.

Question 1
We have $|1+i| = \sqrt{2}$ and $\arg = \pi/4$, so the square roots are the numbers of modulus $\sqrt[4]{2}$ and argument $\pi/8$, and its negative. Note that $2\cos^2\frac{\pi}{8} - 1 = \cos\frac{\pi}{4} = \sqrt{2}/2 = 1 - 2\sin^2\frac{\pi}{8}$ so that

$$\cos\frac{\pi}{8} = \sqrt{\frac{1+\sqrt{2}/2}{2}}, \quad \sin\frac{\pi}{8} = \sqrt{\frac{1-\sqrt{2}/2}{2}}.$$

For $3+4i$ we cannot write the argument explicitly but we can solve

$$(x+iy)^2 = 3+4i \Leftrightarrow x^2 - y^2 = 3, xy = 2$$

either by eyeballing the solution $x = 2, y = 1$, and so the two square roots are $\pm(2+i)$, or by substituting $y = 2/x$ to get $x^2 - 4x^{-2} = 3$ and solve $x^2 = 4$ or $x^2 = -1$, and only the first is possible leading to $x = \pm 2, y = \pm 1$.

Question 2
A function u of class C^2 defined on an open set $U \subset \mathbb{C}$ is harmonic iff $\Delta u := u_{xx} + u_{yy} \equiv 0$. If $f = u + iv$ with u, v real is holomorphic, then the Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$ imply that $u_{xx} = v_{yx} = v_{xy} = -u_{yy}$, where we used the equality of the mixed partial for twice continuously differentiable functions. (Similarly for v.)
Rewrite the function as

$$\frac{1}{2i}\frac{z-\bar{z}}{z\bar{z}} = \frac{1}{2i}\left(\bar{z}^{-1} - z^{-1}\right) = -\frac{1}{2i}\left(z^{-1} - \bar{z}^{-1}\right) = -\mathrm{Im}(z^{-1}) = \mathrm{Re}(iz^{-1})$$

so it is harmonic away from 0 and a conjugate is $\mathrm{Im}(iz^{-1}) = x/(x^2+y^2)$.

Question 3
For (a) the ratio of successive terms is

$$\frac{(2n+1)(2n+2)}{(n+1)^2} z = 4\frac{n+\frac{1}{2}}{n+1} z \to 4z \qquad \text{as } n \to \infty$$

so the ratio test guarantees convergence for $|z| < 1/4$ and divergence for $|z| > 1/4$.
Fof (b) we get as ratio

$$\frac{n^3}{(n+1)^3}\frac{z^3}{3}$$

with guaranteed convergence when $|z| < \sqrt[3]{3}$ and divergence when $|z| > \sqrt[3]{3}$.
When $|z| = \sqrt[3]{3}$, the nth term in the series has modulus n^{-3}, so the series is absolutely (and uniformly) convergent, say by the integral estimation of the sum.

Question 4
It is a geometric series in the expansion parameter (z^2+1), so the condition for convergence is precisely $|z^2+1| < 1$. So z^2 must lie inside the circle of radius 1 centered at -1. The region of convergence is the interior of a figure "8" centered at 0, placed vertically and symmetric about the x- and y-axes. The lines of the 8 are tangent to the diagonals.

Question 5
(a) Parametrise the path by $t \mapsto 1 + it$, $t \in [-1, 1]$. We get

$$\int \frac{dz}{z} = \int_{-1}^{1} \frac{idt}{1+it} = \int_{-1}^{1} \frac{i(1-it)dt}{1+t^2} = \int_{-1}^{1} \frac{idt}{1+t^2} + \int_{-1}^{1} \frac{tdt}{1+t^2} = i \cdot \arctan t|_{-1}^{1} + 0 = \frac{\pi i}{2}$$

the second integral vanishing because the odd function is integrated from -1 to 1.
(b) Parametrise the quarter-circle by $\theta \mapsto \sqrt{2}e^{i\theta}$, $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$. Then the integral is

$$\int \frac{dz}{z} = \int_{-\pi/4}^{\pi/4} \frac{ie^{i\theta}d\theta}{e^{i\theta}} = i\int_{-\pi/4}^{\pi/4} d\theta = \frac{\pi i}{2}.$$

(c) Parametrise the 3/4 circle by $\theta \mapsto e^{-i\theta}$, now $\frac{\pi}{4} \leq \theta \leq \frac{7\pi}{4}$. (Check that this does travel the correct way on the circle.) The same integral now gives $\frac{-3\pi i}{2}$.

By the complex fundamental theorem of calculus, $\int_\gamma f(z)dz$ of a holomorphic function f along a path γ from a to b is equal to $F(b) - F(a)$, if a holomorphic antiderivative F is found in a region containing the path. In our case, $f(z) = z^{-1}$ and we can take Log$z$ as an anti-derivative for cases (a) and (b). However, Log is discontinous along the third path in (c). We can find another branch of log that is holomorphic along the third arc, by making a branch cut along the real axis, say, but in that case, if the value at $1 - i$ is $-\pi i/4$, this log will take at $1 + i$ the non-standard value $-7\pi i/4$, explaining the new answer.