Solutions 9

Question 1

Fifth order pole at $z = i$, so we need the value at $z = i$ of

$$\frac{1}{4!}\frac{d^4}{dz^4}\frac{e^{iz}}{(z+i)^5} = \frac{e^{iz}}{24(z+i)^5}\left(1 + \frac{20i}{z+i} - \frac{180}{(z+i)^2} - \frac{840i}{(z+i)^3} + \frac{1680}{(z+i)^4}\right) = \frac{133}{384i \cdot e}$$

The standard infinite upper half-circle gives

$$\int_0^\infty \frac{\cos x dx}{(x^2+1)^5} = \frac{133}{192}\frac{\pi}{e}.$$

Question 2

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. By the usual upper half-circle method and using the half-residue at zero,

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{1}{4}\int_{-\infty}^\infty \frac{1-\cos 2x}{x^2}dx = \mathrm{Re}\left(\mathrm{PV}\int_{-\infty}^\infty \frac{1-e^{2ix}}{4x^2}dx\right) = \mathrm{Re}\left(\pi i \cdot \mathrm{Res}_{z=0}\frac{1-e^{2iz}}{4z^2}\right) = \pi i \frac{-i}{2} = \frac{\pi}{2}.$$

Question 3 Pretty straightforward following the instructions: the desired integral is half of $\int_{-\infty}^\infty$, and we can replace for simplicity $\cos(px)$ with e^{pix}: the sine integral vanishes because it is odd. The vertical contributions in the rectangle vanish in the $R \to \infty$ limit, because e^{ipz} stays bounded, while $|\cosh z| \to \infty$. The upper integral is $e^{ip\pi}$ times the original (mind that cosh changes sign when adding πi, which takes care of the reversed orientation on the upper side), so all in all we get the desired formula from

$$(1 + e^{ip\pi})\int_0^\infty \frac{\cos px}{\cosh x}dx = \pi i \cdot \mathrm{Res}_{z=\pi i/2}\frac{e^{ipz}}{\cosh z} = \pi i \frac{e^{ip\pi/2}}{\sinh(\pi i/2)} = \pi e^{ip\pi/2}$$

Question 4

By parity, we need to compute $\int_{-\infty}^\infty$ and we use the contour of Q3, but of height i. The upper horizontal side carries a simple pole at $z = i$ so we need to use Cauchy Principal Value on that side,

$$\mathrm{PV}\int_{-\infty}^\infty \frac{z+i}{\sinh \pi z}dz = \mathrm{PV}\int_{-\infty}^\infty \frac{z}{\sinh \pi z}dz = \int_{-\infty}^\infty \frac{z}{\sinh \pi z}dz,$$

the first equality because $\sinh \pi z$ is odd (the second because there is no pole left). So the half-residue formula gives

$$\mathrm{PV}\oint = 2\int_{-\infty}^\infty \frac{x dx}{\sinh \pi x} = \pi i \cdot \mathrm{Res}_{z=i}\frac{z}{\sinh \pi z} = \pi i \cdot \frac{i}{-\pi} = 1,$$

whence we get $1/4$ for the original integral.

Question 5

Convert to a complex integral around the unit circle $z = e^{i\theta}$ with $dz = ie^{i\theta}d\theta$ and $2i\sin\theta = z + z^{-1}$:

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} = \int_C \frac{-iz^{-1}}{5 - 2iz + 2iz^{-1}}dz = \int_C \frac{dz}{2z^2 + 5iz - 2}.$$

The denominator vanishes at $z = -5i/4 \pm 3i/4$ of which $-i/2$ is inside the unit circle. The residue there is the value at $-i/2$ of

$$\frac{z + i/2}{2z^2 + 5iz - 2} = \frac{1}{2z + 4i}$$

which is $1/3i$, so the integral is $2\pi/3$.

Question 6
Use a keyhole contour, making a cut along the positive real axis so that $z^\alpha = x^\alpha$ on the upper side and $z^\alpha = \exp(2\pi i\alpha)x^\alpha$ on the lower side. The two integrals along the positive real axis add up to

$$(1-\exp(2\pi i\alpha))\int_0^\infty \frac{x^\alpha\,dx}{(x^2+1)^2} = -2i\exp(\pi i\alpha)\sin(\pi\alpha)\int_0^\infty \frac{x^\alpha\,dx}{(x^2+1)^2}.$$

On the circle of small radius r, the integral is bounded by $2\pi r\cdot r^\alpha/(1-r^2)^2$ which clearly goes to 0 with r when $\alpha > -1$. On a large circle, we have a bound $2\pi R\cdot R^\alpha/(R^2-1)^2$ which again goes to 0 with $R\to\infty$ as long as $\alpha < 3$. So in the limit the only contributions come from the real line as detailed above.
The singularities are at $z=\pm i$; they are double poles. The residues are

$$\frac{d}{dz}\frac{z^\alpha}{(z+i)^2}\bigg|_{z=i} = \frac{\alpha i^{\alpha-1}}{(2i)^2} - \frac{i^\alpha}{(2i)^3} = \frac{(\alpha-1)ie^{\pi i\alpha/2}}{4},$$

and

$$\frac{d}{dz}\frac{z^\alpha}{(z-i)^2}\bigg|_{z=-i} = \frac{\alpha(-i)^{\alpha-1}}{(-2i)^2} - \frac{(-i)^\alpha}{(-2i)^3} = -\frac{(\alpha-1)ie^{3\pi i\alpha/2}}{4},$$

summing to

$$\frac{i(\alpha-1)}{4}\exp(\pi i\alpha)\left(-2i\sin(\pi\alpha/2)\right) = \frac{(\alpha-1)}{2}\exp(\pi i\alpha)\sin(\pi\alpha/2).$$

The residue formula gives

$$-2i\exp(\pi i\alpha)\sin(\pi\alpha)\int_0^\infty \frac{x^\alpha\,dx}{(x^2+1)^2} = \pi i(\alpha-1)\exp(\pi i\alpha)\sin(\pi\alpha/2),$$

$$\cos(\pi\alpha/2)\int_0^\infty \frac{x^\alpha\,dx}{(x^2+1)^2} = \frac{\pi(1-\alpha)}{4}\sin(\pi\alpha/2).$$

Remark: If you exploit the symmetry $x\leftrightarrow(-x)$ of the denominator, you can integrate instead on an upper half circle, and get a way with a single residue calculation at $z=+i$. But the more residues, the merrier!

Question 7
We integrate $\dfrac{z\log z\,dz}{z^3+z^2+z+1}$ on the keyhole contour, using a tiny circle of radius r about 0 and a large outside circle of radius R. The large circle integral is bounded by $C\cdot 2\pi R\cdot(R\log R)/R^3$ for some constant $C<1$, as soon as R is large, so it goes to 0 as $R\to\infty$. The small circle integral is bounded by $c\cdot r^2\log r$ for some constant c, and it similarly goes to 1 as $r\to 0$. The two integrals on the real axis combine to

$$-(2\pi i)\int_0^\infty \frac{xdx}{x^3+x^2+x+1}$$

because the upper value of the denominator is $x\log x$ and the lower one is $x(\log x+2\pi i)$. So by the residue formula, our integral is the negative of the sum of the three residues of $\dfrac{z\log z}{z^3+z^2+z+1}$ at the points $z+\pm i$ and $z=-1$. These are:

$$-\frac{\pi}{2\cdot 2i(i+1)},\quad -\frac{\pi i}{2},\quad \frac{3\pi}{2\cdot(-2i)(-i+1)}$$

so the integral is $\pi/4$.

Question 8
We can get a more precise result. Take $f(z) = (z-1)^n e^z$, $g(z) = -a$ and consider the circle of radius 1 centered at 1, where $|f| \geq 1$ but $|g| < 1$. So f and $f+g$ have the same number of roots in that disk, with multiplicities counted, and than number for f is clearly n. Now $f+g$ cannot have multiple roots: vanishing of its derivative requires $n(z-1)^{n-1}e^z + (z-1)^n e^z = 0$, or $(z-1)^{n-1}(z+n-1) = 0$. Now $z=1$ is not a root of $(z-1)^n e^z = a$ for $a \neq 0$, and $z = 1-n$ is not inside the disk so is also not a root.

Question 9
Choosing the function to be real-valued on the upper edge of the keyhole contour, it gets multiplied by $e^{-2\pi i p}$ on the lower edge and we get from the half-residue formula

$$(1 - e^{-2\pi i p})\mathrm{PV}\int_0^\infty \frac{x^{-p}dx}{x-1} = \pi i \cdot (\mathrm{Res}_{1^+} + \mathrm{Res}_{1^-})\,\frac{z^{-p}}{z-1}$$

where the $\pm$ superscripts indicate we consider the upper, rest. the lower branch of the function. The residues are 1 and $e^{-2\pi i p}$, so we get

$$\mathrm{PV}\int_0^\infty \frac{x^{-p}dx}{x-1} = \pi i \frac{1+e^{-2\pi i p}}{1-e^{-2\pi i p}} = \pi\cot(\pi p)$$

Question 10
The function can be made single-valued in the region enclosed by the contour indicated (the effect of the straight lines and tiny circles equals that of a branch cut). We choose the holomorphic function which is real-valued on the upper side of the interval $[0,1]$; its value on the lower side is multiplied by $\exp(4\pi i/3)$. Cauchy's theorem gives

$$(1 - e^{4\pi i/3})\int_0^1 \frac{dx}{\sqrt[3]{x^2 - x^3}} = \oint_{C_R} \frac{dz}{\sqrt[3]{z^2 - z^3}} = e^{-\pi i/3}\oint_{C_R} \frac{dz}{\sqrt[3]{z^3 - z^2}} = e^{-\pi i/3}\oint_{C_R} \frac{dz}{z\sqrt[3]{1 - 1/z}}$$

having used the cube root which is real on the (large) positive real axis. (The factor $\exp(-\pi i/3)$ arises by tracing the moving from $1-\varepsilon$ to $1+\varepsilon$ on a tiny circle in the upper half plane.)
As $R \to \infty$, the last integral converges to $1\pi i$ and we get as the answer

$$\frac{2\pi i \cdot e^{-\pi i/3}}{1 - e^{4\pi i/3}} = \frac{2\pi}{\sqrt{3}}.$$