Solutions 8

Question 1 Because of the high power in the denominator, this requires Cauchy's formula for derivatives, specifically the second derivative of e^{iz} at $z = 0$. The answer is $2\pi i/2!(-1) = -\pi i$.

Question 2
There are two singularities inside the circle, at $z = \pm i$ each of which we can isolate with a little circle. The factorization $(z^2+1)^2 = (z+i)^2(z-i)^2$ tells us that we will need again to use Cauchy's formula for derivatives, the first derivative in this case. The singularities contributes the value of the derivative of the derivative of $e^{zt}/(z \pm i)^2$ at $z = \pm i$. The contributions are $\frac{i}{4}(it \mp 1)e^{\pm it}$, and summing them gives $(\sin t - t\cos t)/2$.

Question 3
Cauchy's formula for derivatives, or the residue formula for $z = -1$, gives

$$\frac{1}{2!}\frac{d^2}{dz^2}\left(ze^{tz}\right)\Big|_{z=-1} = \frac{1}{2}\left(2te^{tz} + t^2 ze^{tz}\right)\Big|_{z=-1} = te^{-t} - \frac{t^2}{2}e^{-t}$$

Question 4
We proved in class the existence of Laurent series expansion for an annulus, by expanding the two Cauchy integrals into geometric series: if a function f is holomorphic in an annulus $r < |z| < R$, the outer circle integral becomes a Taylor series convergent for $|z| < R$ and the inner circle integral becomes a Laurent principal part convergent for $|z| > r$. (See also Sarason, VIII.6.) Thus, the negative part outside the disk $|z| < r$; but for an isolated singularity, we can take r arbitrarily small so it converges on $\mathbf{C} \setminus \{0\}$.

Question 5
We have $\sin(z + z^{-1}) = \sin z \cos z^{-1} + \sin z^{-1}\cos z$ and get for each term the product of series expansions

$$\sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!} \cdot \sum_{n=0}^{\infty}(-1)^n \frac{z^{-2n}}{(2n)!} \quad \text{and} \quad \sum_{n=0}^{\infty}(-1)^n \frac{z^{-2n-1}}{(2n+1)!} \cdot \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}$$

The z^{-1} coefficient in the two series are

$$\sum_{n=0}^{\infty}(-1)^n(-1)^{n+1}\frac{1}{(2n+1)!}\frac{1}{(2n+2)!} \quad \text{and} \quad \sum_{n=0}^{\infty}(-1)^n(-1)^n\frac{1}{(2n+1)!}\frac{1}{(2n)!}$$

and so the residue is given by the (very rapidly convergent) series

$$\sum_{n=0}^{\infty}\left(\frac{1}{(2n)!(2n+1)!} - \frac{1}{(2n+1)!(2n+2)!}\right)$$

Question 6

$$(z^2-1)\sin(1/z^2) = (z^2-1)\sum_{n=0}^{\infty}(-1)^n\frac{z^{-4n+2}}{(2n-1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{-4n+4} - z^{-4n+2}}{(2n-1)!} = \sum_{k=0}^{\infty} f_{2k}z^{-2k},$$

where $f_{2k} = -(-1)^{k/2}/(k+1)!$ if k is even and $f_{2k} = (-1)^{(k-1)/2}/k!$ if k is odd.

Question 7
First annulus: $0 < |z| < 1$, we expand $(1-z)$ in the denominator in a geometric series to get

$$f(z) = \sum_{n \geq 0} z^{n-2}.$$

Second annulus, $1 < |z| < \infty$, we have

$$\frac{1}{z^2(1-z)} = \frac{-1}{z^3(1-z^{-1})} = -\sum_{n \geq 0} z^{-n-3}$$

This "converges" also at $z = \infty$ (and we get the value 0).

Question 8
Partial fractions is the easiest way here, $\frac{z}{(z-1)(2-z)} = \frac{1}{z-1} - \frac{2}{z-2}$, followed by the relevant geometric series expansions.
(a) $\sum_{n=0}^{\infty} (2^{-n} - 1) z^n$. (b) $\sum_{n=-\infty}^{-1} z^n + \sum_{n=0}^{\infty} 2^{-n} z^n$ (c) $\sum_{n=-\infty}^{-1} (1 - 2^{-n}) z^n$
(d) Set $z - 1 = w$ and expand

$$\frac{1}{z-1} - \frac{2}{z-2} = \frac{1}{w} - \frac{2}{w-1} = \frac{1}{w} - 2\sum_{n=-\infty}^{-1} w^n = (z-1)^{-1} - 2\sum_{n=-\infty}^{-1} (z-1)^n$$

(e) Set $w = z - 2$ and expand

$$\frac{1}{z-1} - \frac{2}{z-2} = \frac{1}{w+1} - \frac{2}{w} = 2w^{-1} + \sum_{n=0}^{\infty} (-1)^n w^n = 2(z-2)^{-1} + \sum_{n=0}^{\infty} (-1)^n (z-2)^n$$

Question 9
(a) Simple poles at qth roots of unity $\exp(2\pi i k/q)$, residues $-\exp(2\pi i k(p+1))/q$.
(b) Double poles at $z = \pm 1$, residues 1 at both.
(c) Simple poles at $z = \exp(\pm 2\pi i/3)$, residues

$$\frac{\cos(\exp(\pm 2\pi i/3))}{2\exp(\pm 2\pi i/3) + 1} = \mp i \frac{\cos(\exp(\pm 2\pi i/3))}{\sqrt{3}}.$$

Question 10
As in the hint, consider the square S_N with vertices at the four points $(N+1/2)(\pm 1 \pm i)$, with the four choices of sign. Now $|\cos(x+iy)|^2 = \cos^2 x + \sinh^2 y$ and $|\sin(x+iy)|^2 = \sin^2 x + \sinh^2 y$, so on the vertical sides, $|\cot(\pi z)| < 1$ while on the horizontal ones, $|\cot(\pi z)| < 1 + 1/\sinh^2(N+1/2)$ which is bounded by 2 for $N \geq 1$. So $|f(z)\cot(\pi z)| < (\text{const.})/N^2$ on S_N for large N, and

$$\int_{S_N} f(z)\cot(\pi z) dz \to 0 \quad \text{as} \quad N \to \infty.$$

For N large enough to enclose all poles of f, the sum of the residues inside S_N is the contour integral.
The singularities come from those of f and those of $\cot(\pi z)$ — the integers. If f does not have a pole at the integer n, then $f(z)\cot(\pi z)$ has a simple pole, and from Taylor expansion of $\sin, \cos$ we see that

$$f(z)\cot(\pi z) = \frac{f(z)}{\pi(z-n)} + (\text{holomorphic})$$

and so the residue of $f(z)\cot(\pi z)$ at $z = n$ is $f(n)/\pi$. On the other hand, at any pole p of f (integer of not) we get the residue $\mathrm{Res}_p\,[f(z)\cot(\pi z)]$. Setting the sum of the residues to zero shows that

$$\left(\sum_{-N}^{N} f(n) + \pi \sum_{p\in S_N} \mathrm{Res}_p\,[f(z)\cot(\pi z)]\right) \to 0 \quad \text{as} \quad N \to \infty$$

as desired.

To get Euler's expansion of $1/\sin^2 w$, notice that, when $w \notin \mathbb{Z}$

$$\mathrm{Res}_{z=\pi w}\frac{\cot(\pi z)}{(z-\pi w)^2} = \frac{d}{dz}\cot(\pi z)\bigg|_{z=\pi w} = \frac{\pi}{\sin^2(\pi w)}.$$

Question 11

Integrating the left side once leads to $\cot z$. The right-hand side can also be integrated term-by term, because of uniform convergence away from the poles, and gives

$$\cot z = \frac{1}{z} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(\frac{1}{z-n\pi} - \frac{1}{n\pi}\right) + A = \frac{1}{z} + \sum_{1}^{\infty}\frac{2z}{z^2 - n^2\pi^2} + A$$

Notice the 'counterterm' $1/n\pi$, needed for convergence; integrating term-by-term from any z_0 gives $1/(z-n\pi) - 1/(z_0 - n\pi)$, and we have set $z_0 = 0$ in all terms, save when $n = 0$. This slightly inconsistent choice is leaving an ambiguous additive constant. In the last term, we have combined terms with opposite signs of n. We can integrate again and get

$$\log\sin z = \log z + \sum_{1}^{\infty}\left(\log(z^2 - n^2\pi^2) - \log(-n^2\pi^2)\right) + Az + B.$$

Both sides are multi-valued, but miraculously (or not) there is only additive ambiguity in the form of integer multiples of $2\pi i$. So the exponentials are single-valued and lead to the identity in the problem, modulo a factor of $e^B e^{Az}$. Rule out the e^{Az} term by noticing that both sides are odd functions of z, and check the value and the derivative at $z = 0$ to find that $B = 0$.