Solutions 7

Question 1

Let $p(z)$ be a non-constant polynomial. If $p(z) \neq 0$ for all $z \in \mathbb{C}$, then the function $1/p(z)$ is holomorphic. In particular, it is continuous. Let $p(0) = a$, say. For a large enough $R > 0$, $1/|p(z)| < |a|$ for $|z| \geq R$. Restricting to the closed disk of radius R, the continuous function $1/|p(z)|$ will achieve its maximum value somewhere; this cannot be on the boundary, since it takes a greater value $|a|$ at an interior point. So it achieves its maximum at some interior point, which contradicts the maximum principle.

Question 2

For every $q \in E$, some disk around q is also contained in E, and then q can be connected to any point in that disk by a straight line segment that lies in E. So if q can/cannot be connected to p by a continuous or polygonal path, then the same property holds for each point in a neighborhood of q.

So if E is connected then it is polygonally path connected, else $E = C_p \coprod N_p$.

Conversely, if E is path connected then it is connected: else, $E = E_1 \coprod E_2$ with both open and not empty, so a path $\gamma : [0, 1] \to E$ joining $\gamma(0) \in E_1$ with $\gamma(1) \in E_2$ would decompose $[0, 1]$ into two disjoint subsets $\gamma^{-1}(E_{1,2})$. Now let x be the least upper bound of $\gamma^{-1}(E_1)$. Whichever of the sets x lies in, we get a contradiction with the continuity of γ, because there are points arbitrarily close to x mapping to the other set (all points above x map to E_2 and some points below, but arbitrarily close to x map to E_1).

Question 3

The idea is simple, the point $(0, 0)$ cannot be joined to $(1/\pi, 0)$ because the obvious path that does the job, the graph of the function, is discontinuous at 0. But a proof of impossibility requires a bit more. Let $\gamma : [0, 1] \to G$ be our path (G is our set) and let $C \subset [0, 1]$ be the closed subset mapping to the vertical segment $x = 0, y \in [-1, 1]$. This contains maximal number $a < 1$. (C is closed and bounded so it contains its least upper bound.)

We'll show γ is discontinuous at a by finding a sequence $t_n \to a$ with the y-component $y(\gamma(t_n))$ converging to any prescribed value $v \in [-1, 1]$. Indeed, $\delta := x(\gamma(a + \varepsilon)) > 0$ for small $\varepsilon > 0$, so by the intermediate value theorem the values of $x(\gamma(t))$ cover the interval $(0, \delta)$, and we can find a time $t_1 < a + \varepsilon$ with $y(\gamma(t_1)) = \sin(1/x(\gamma(t_1))) = v$. Repeat now with $\varepsilon/2^n$ to get your sequence.

Question 4 (5.48)

No. Liouville's theorem applies to analytic functions over $\mathbb{C}$, not just over $\mathbb{R}$. While we can extend $\sin x$ to a holomorphic function of all complex values z, it will not be bounded (growing exponentially in the imaginary directions).

Question 5 (5.49)

The rectangle is closed. If the function was analytic, it would be continuous, in particular $F(z)$ would achieve a maximum and hence would be bounded. Periodicity allows you to extend it to a bounded holomorphic function on all of $\mathbb{C}$, contradicting Liouville's theorem.

Question 6 (5.55)

As in the hint, integrate $\operatorname{Re}\ln(1 + z)$ on the circle C of radius 1. There is a trouble point at $z = -1$, but we can take the limit of $\ln(1 + rz)$ for $r < 1$ under the integral sign.

$$\oint_C \operatorname{Re}\ln(1 + z) = \operatorname{Re}\ln(1) = 0$$

but parametrizing we get

$$\int_0^{2\pi} \mathrm{Re}\ln(1+\cos\theta+i\sin\theta)d\theta = \int_0^{2\pi} \ln(2+2\cos\theta)d\theta = 2\pi\ln 2 + \int_0^{2\pi} \ln(2\cos^2\frac{\theta}{2})d\theta =$$
$$= 4\pi\ln 2 + \int_0^{2\pi} \ln(\cos^2\frac{\theta}{2})d\theta = 4\pi\ln 2 + 4\int_0^{\pi} \ln(|\cos\theta|)d\theta = 4\pi\ln 2 + 4\int_0^{\pi}\ln(\sin\theta)d\theta$$

giving the formula.

Question 7 (5.85)
Integrate $\oint e^z/zdz$ around the unit circle using Cauchy's formula, and take real and imaginary parts; remember that $dz/z = id\theta$. I think the answers in the book are switched.

Question 8
Fix z_0 not on γ, and necessarily some distance $d > 0$ away from it. Let z be any point in the disk $|z-z_0| < d$. Write $\zeta - z = (\zeta - z_0) - (z - z_0)$,

$$\int_\gamma \frac{\varphi(\zeta)d\zeta}{(\zeta-z_0)-(z-z_0)} = \int_\gamma \frac{\varphi(\zeta)}{1-(z-z_0)/(\zeta-z_0)}\frac{d\zeta}{\zeta-z_0} =$$
$$= \int_\gamma \sum_{n=0}^{\infty}\left(\frac{z-z_0}{\zeta-z_0}\right)^n \varphi(\zeta)\frac{d\zeta}{\zeta-z_0} = \sum_{n=0}^{\infty}(z-z_0)^n \int_\gamma \frac{\varphi(\zeta)d\zeta}{(\zeta-z_0)^{n+1}}$$

where we integrate the series term-by-term thanks to its uniform convergence in any closed disk around z_0 of radius less than d.
For large z, the integrand is continuous in ζ and converges uniformly to 0 as $z \to \infty$, so the integral (over the compact interval parametrising γ) also goes to zero.

Question 9

1. If γ is a (simple) closed curve, the integral gives zero when z is outside and $\varphi(z)$ when z is inside, the first from Cauchy's theorem and the second from Cauchy's formula.
2. Choose γ to be the straight line segment S from a to b. Then, I claim that

$$\int_a^b \frac{d\zeta}{\zeta - z} = \mathrm{Log}\frac{z-b}{z-a} \tag{1}$$

   Indeed, $\mathrm{Log}(z-\zeta)$ is an antiderivative (in ζ) of $(\zeta-z)^{-1}$ in the region $\zeta \in \mathbf{C}$ where $z-\zeta$ is not on the negative real axis, for example when z has a large real part; and we can then use the complex fundamental theorem of calculus. Noting that the function in (1) is defined and holomorphic in z, as long as z avoids S, the identity principle then tells us that (1) must hold for all z off S.

   When deforming γ to a general curve, describing the correct value of log at a general point is more difficult: imagine for instance γ spiraling in many times around b before reaching it. One answer is that, far away from the axis ab, we must use the same formula (1), and extend continuously (and holomorphically) to the complement of γ. A more precise description requires the notion of winding numbers ...

Question 10

1. If $f = u + iv$, we have $g(x + iy) = \overline{f(x - iy)} = u(x, -y) - iv(x, -y)$ and so

$$\frac{\partial \mathrm{Re}\, g}{\partial x} = \frac{\partial u}{\partial x}, \quad \frac{\partial \mathrm{Im}\, g}{\partial x} = -\frac{\partial v}{\partial x}, \quad \frac{\partial \mathrm{Re}\, g}{\partial y} = -\frac{\partial u}{\partial y}, \quad \frac{\partial \mathrm{Im}\, g}{\partial y} = \frac{\partial v}{\partial y},$$

and Cauchy-Riemann for g follows from the same for f. (Real differentiability is clear).
2. The assumption implies that $f - g$ vanishes identically on the real axis; the identity principle then says that $f - g \equiv 0$ in E.

Question 11

From the previous question we know that the extended f is holomorphic, except possibly on the real axis, if we have not assumed continuous differentiability. Morera's theorem tells us to check the vanishing of $\int f(z)dz$ around any rectangle R with sides parallel to the axes. If R lies completely above or completely below the real axis, vanishing of the integral checks out by Cauchy. Note now that vanishing also holds when R has one side on the real axis, because we can view R as a limit of rectangles that just avoid the axis, and pass to the limit in the integral by continuity of f. Finally, assume that R covers the real axis, and split it into two rectangles, one above and one below. Then, $\int f$ vanishes on each of these subrectangles, but the sum of the two is $\int_R f$ because the real edge cancels out in the sum of the integrals.