# Solutions 6

**Question 1**

$$\oint \bar{z}dz = \oint (x-iy)(dx+idy) = \oint \{(x-iy)dx + (y+ix)dy\} = \iint_D (i+i)dxdy = 2i\cdot \mathrm{Area}(D).$$

Without differentials, invoke the complex form of Green's theorem

$$\oint fdz = 2i\cdot \iint \frac{\partial f}{\partial \bar{z}}dxdy, \quad \text{and } \frac{\partial \bar{z}}{\partial \bar{z}} = 1.$$

**Question 2**

As $z$ ranges over the arc of angle $\pi/8$, $z^2$ ranges over the doubled arc so its real part is bounded below by $R^2/\sqrt{2}$; the integrand therefore has quadratic-exponential decay and the integral vanishes, the arc length being $\pi R/8$.

The integral on the real line converges to $\frac{1}{2}\sqrt{\pi}$. Parametrising $z = 2^{1/4}t(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8})$ gives the contribution of the ray as

$$-2^{1/4}(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8})\int_0^\infty e^{-t^2}(\cos t^2 - i\sin t^2)dt.$$

Since the contour integral vanishes by Cauchy, we get that

$$\frac{1}{2}\sqrt{\pi}(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8})\cdot 2^{-1/4} = \int_0^\infty e^{-t^2}(\cos t^2 - i\sin t^2)dt.$$

With $\cos\frac{\pi}{8} = 2^{-3/4}\sqrt{\sqrt{2}+1}$, we get the advertised formula.

**Question 3**

As before, the real half-axis contributes $\frac{1}{2}\sqrt{\pi}$. We parametrize the ray of argument $\pi/4$ by $z = t(1+i)/\sqrt{2}$ and get form the vanishing of the contour integral

$$\frac{(1+i)}{\sqrt{2}}\int_0^\infty (\cos t^2 - i\sin t^2)dt = \frac{1}{2}\sqrt{\pi}$$

whence

$$\int_0^\infty \cos t^2 dt = \int_0^\infty \sin t^2 dt = \sqrt{\frac{\pi}{8}}$$

assuming that the arc contribution to the integral vanishes in the $R \to \infty$ limit.

Now that vanishing is a bit subtle, because the integrand decays exponentially on most of the arc, but not near the $\pi/4$ ray. The point is that it fails to decay on a narrow enough sliver. Subdivide the range $[0, \pi/4]$ of the argument into $[0, \pi/4-\varepsilon]$ and $[\pi/4-\varepsilon, \pi/4]$, with $\varepsilon$ to be fixed later. On the former, the integrand $e^{-z^2}$ is bounded in modulus by $e^{-R^2\cos\pi/2-2\varepsilon} = e^{-R^2\sin 2\varepsilon}$. This is overestimated by $e^{-R^2\varepsilon}$, for small $\varepsilon$. The remaining arc has length $R\varepsilon$ and the integrand is bounded by 1 in absolute value. Overall, we get for the arc integral the bound

$$\frac{\pi R}{4}\exp(-R^2\varepsilon) + R\varepsilon.$$

Choosing now $\varepsilon = R^{-3/2}$ gives the bound $\frac{\pi R}{4}e^{-\sqrt{R}} + R^{-1/2}$, which goes to 0 as $R \to \infty$.

**Question 4**
We apply Cauchy's theorem to a contour $C_R$ consisting of the interval $[-R, R]$ and the upper half-circle of radius $R$. This encloses (for large $R$) the two points $z = \pm 1 + i$ where the integrant fails to be holomorphic. We also use the integrand $e^{iz}/(4+z^4)$, which has the benefit of decaying on the half-circle as $R \to \infty$ because $|e^{iz}| \leq 1$. We can isolate the two bad points by two small circles $C_\pm$ as in the proof of Cauchy's formula, or else split up the integral into the two quarter-circles (noting that the imaginary axis is traversed twice in opposite directions so its contribution cancels out); Cauchy's theorem implies

$$\oint_{C_R} \frac{e^{iz}dz}{z^4+4} = \oint_{C_+} \frac{e^{iz}dz}{z^4+4} + \oint_{C_-} \frac{e^{iz}dz}{z^4+4}.$$

For the $C_+$ contribution we split the denominator as $(z-1-i)(z+1+i)(z^2+2i)$; for $C_-$ as $(z+1-i)(z-1+i)(z^2-2i)$. Cauchy gives

$$\oint_{C_+} \frac{e^{iz}dz}{z^4+4} = \oint_{C_+} \frac{e^{iz}}{(z+1+i)(z^2+2i)} \frac{dz}{(z-1-i)} = 2\pi i \frac{e^{i-1}}{(2+2i)4i} = \frac{\pi}{8e}(1-i)(\cos 1 + i \sin 1);$$

$$\oint_{C_-} \frac{e^{iz}dz}{z^4+4} = \oint_{C_-} \frac{e^{iz}}{(z-1+i)(z^2-2i)} \frac{dz}{(z+1-i)} = 2\pi i \frac{e^{-i-1}}{(-2+2i)(-4i)} = \frac{\pi}{8e}(1+i)(\cos 1 - i \sin 1);$$

Their sum is $\frac{\pi}{4e}(\cos 1 + \sin 1)$. On the other hand,

$$\oint_{C_R} \frac{e^{iz}dz}{z^4+4} = \int_{-R}^{R} \frac{e^{ix}dx}{x^4+1} + iR \int_0^\pi \frac{\exp(Re^{it})dt}{R^4e^{4it}+4}$$

The second integrand can be bounded by $(R^4-4)^{-1}$ so the second term is bounded by $\pi R/(R^4-4)$ and goes to 0 as $R \to \infty$. On the other hand, the first term goes to

$$\int_{-\infty}^{\infty} \frac{\cos x dx}{x^4+4} + i \int_{-\infty}^{\infty} \frac{\sin x dx}{x^4+4}.$$

This must equal $\frac{\pi}{4e}(\cos 1 + \sin 1)$, so that is the value of the cosine integral — twice the integral in the question — and the sine integral vanishes (which also follows because it is an odd function).

**Question 5**
For (a), the circle of radius 4 centered at 1 includes the singularity $z = \pi i$ so Cauchy tells us the answer is $e^{3\pi i} = -1 \cdot 2\pi i$. For (b), we have $\sqrt{\pi^2+4} + \sqrt{\pi^2+4} = 2\sqrt{\pi^2+4} > 6$ so the function is holomorphic inside the ellipse and we get 0.

**Question 6**
The problem points are at $z = \pm 1$. The first rectangle includes both, the second only the point $z = +1$. Find the contributions at the two points by factoring $z^2 - 1 = (z-1)(z+1)$ as

$$\frac{\cos \pi}{2} = -\frac{1}{2}, \quad \text{respectively} \quad \frac{\cos(-\pi)}{-2} = \frac{1}{2}$$

So the answers are (a) 0 and (b) $-\pi i$, from $+1$ only.

**Question 7**
The integral of $ze^{iz}/(z^4+4)$ on a large half-circle is bounded by $2\pi R/R^4 = 2\pi R^{-3}$ and vanishes in the $R \to \infty$ limit. (Choose $R^4 > 8$ large so that $|z^4+4| \geq R^4/2$.) There are two problem points in the upper half-plane, at $z = 1+i$ and $z = -1+i$. We factor

$$\frac{ze^{iz}}{z^4+4} = \frac{ze^{iz}}{(z-(1+i))(z+1+i)(z^2+2i)} = \frac{ze^{iz}}{(z-(-1+i))(z-1+i)(z^2-2i)},$$

giving contributions to the Cauchy formula

$$\frac{\exp(i(1+i))}{2\cdot 4i} = \frac{\cos 1 + i\sin 1}{8ie} \quad \text{at } z = 1+i \quad \text{and} \quad \frac{\exp(i(-1+i))}{2\cdot(-4i)} = -\frac{\cos 1 - i\sin 1}{8ie} \quad \text{at } z = 1+i,$$

so the contributions add up to give the formula

$$\int_0^\infty \frac{x\sin x dx}{x^4+4} = \frac{1}{2}\int_{-\infty}^\infty \frac{x\sin x dx}{x^4+4} = \pi\cdot\frac{\cos 1 + i\sin 1 - \cos 1 + i\sin 1}{8ie} = \frac{\pi\sin 1}{4e}$$

**Question 8**

The integration contour $Q_R$ in both cases now consists of the real interval $[0,R]$, the first quarter-circle of radius $R$, and the imaginary interval $[0,iR]$ traveled backwards. There is only one bad point at $z = a(1+i)\sqrt{2}$ and writing the integrand as

$$\frac{dz}{z^4+a^4} = \frac{1}{(z^2+ia^2)(z+(1+i)a/\sqrt{2}}\frac{dz}{z-(1+i)a/\sqrt{2}}$$

gives a Cauchy contribution of $2\pi i/2ia^2\cdot\sqrt{2}(1+i)a$. The straight line integrals give

$$(1-i)\int_0^R \frac{dx}{x^4+a^4}$$

while the circle integral is shown to vanish in the $R\to\infty$ limit by the usual argument. So the answer is

$$\int_0^\infty \frac{dx}{x^4+a^4} = \frac{2\pi i}{2ia^3\cdot 2\sqrt{2}} = \frac{\pi}{a^3 2\sqrt{2}}.$$

The second integral works similarly but now the Cauchy contribution at $z = (1+i)a/\sqrt{2}$ is

$$2\pi i\frac{(1+i)a/\sqrt{2}}{2ia^3\cdot\sqrt{2}(1+i)} = 2\pi/4a^2 = \pi/2a^2$$

and this is now twice the desired integral, because

$$\int_0^\infty \frac{(iy)d(iy)}{y^4+a^4} = -\int_0^\infty \frac{(x)dx}{x^4+a^4}$$

and this appears with a minus sign in the contour integral, therefore

$$\lim_{R\to\infty}\oint_{Q_R}\frac{zdz}{z^4+z^4} = 2\int_0^\infty \frac{xdx}{x^4+a^4}$$

**Question 9**

For $\alpha = m/n$, $z\mapsto z^\alpha$ is a composition of $z\mapsto w = z^m$ and $w\mapsto w^{1/n}$, both of which are holomorphic (the second as the inverse of a holomorphic function with non-vanishing derivative. So the composition is holomorphic. The chain rule gives

$$\frac{d}{dz}(z^\alpha) = \frac{d}{dz}(w^{1/n}) = \frac{1}{n}w^{(1-n)/n}\frac{dw}{dz} = \frac{1}{n}w^{(1-n)/n}mz^{m-1} = \frac{m}{n}z^{m-1-m(n-1)/n} = \frac{m}{n}z^{m/n-1} = \alpha z^{\alpha-1}$$

as desired.

For real $\alpha$, to check holomorphy without invoking the $\exp(\alpha \mathrm{Log} z)$ definition, it is easiest to use the polar form of the CR equations (HW2 Q6):

$$r\frac{\partial\left(r^\alpha \cos(\alpha\theta)\right)}{\partial r} = \alpha r^\alpha \cos(\alpha\theta) = \frac{\partial\left(r^\alpha \sin(\alpha\theta)\right)}{\partial\theta},$$
$$\frac{\partial\left(r^\alpha \cos(\alpha\theta)\right)}{\partial\theta} = -\alpha r^\alpha \sin(\alpha\theta) = r\frac{\partial\left(r^\alpha \sin(\alpha\theta)\right)}{\partial r};$$

this, and continuous real-differentiability, imply the holomorphy of the directly defined $z^\alpha$. To check the identity $d(z^\alpha)/dz = \alpha z^{\alpha-1}$ we can just restrict to real $z$, where we know the formula, and the identity principle then extends it to $\mathbf{C}$ minus the negative real axis.
Since we can keep differentiating, the $k$th derivative is $\frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}z^{\alpha-k}$. The Taylor series of the function around $z = 1$ is then as written, and converges on any disk centered at 1 on which the function is holomorphic, in particular, on the disk of radius 1.
(That is in fact the disk of convergence whenever $\alpha$ is not an integer; the function stops being holomorphic at $z = 0$.)