Solutions 5

Question 1

As in the hint, if $a(z) = \sum a_n z^n \equiv 0$, then $a(0) = a'(0) = a''(0) = \cdots = 0$. But up to factorials, these are the coefficients a_n. The key point is that we can differentiate the series term-by-term.

Question 2

The ratio of successive terms in the series is $z^2/(-4)(n+1)(n+k+1)$; for any z, it goes to 0 with n, thus guaranteeing the infinite radius of convergence. We can safely differentiate term-by term to get

$$zJ_k'(z) = \sum_{n=0}^{\infty}(k+2n)(-1)^n \frac{(z/2)^{k+2n}}{n!(n+k)!}$$
$$z^2 J_k''(z) = \sum_{n=0}^{\infty}(k+2n)(k+2n-1)(-1)^n \frac{(z/2)^{k+2n}}{n!(n+k)!}$$

Renumbering for convenience $n \mapsto n-1$ the terms in $z^2 J_k(z)$ gives

$$\sum_{n=0}^{\infty} 4n(n+k)(-1)^{n-1} \frac{(z/2)^{k+2n}}{n!(n+k)!}$$

and the three series for $J, zJ', z^2 J''$ sum term-by-term to $\sum_{n=0}^{\infty}(-1)^n k^2 \left(\frac{z}{2}\right)^{k+2n} / n!(n+k)!$, giving the differential equation.

Second Question 2

$(1 - z + z^2) = (1 + \omega z)(1 + \bar{\omega} z)$ where $\omega = (-1 + i\sqrt{3})/2$. So

$$\frac{1}{1 - z + z^2} = \frac{1}{i\sqrt{3}} \left(\frac{\omega}{1 + \omega z} - \frac{\bar{\omega}}{1 + \bar{\omega} z} \right)$$

So we get for the coefficient of z^n in the expansion

$$\frac{(-1)^n}{i\sqrt{3}}(\omega^{n+1} - \bar{\omega}^{n+1}) = \frac{(-1)^n}{\sqrt{3}} 2 \sin \frac{2\pi(n+1)}{3}$$

giving the sequence $1, 1, 0, -1, -1, 0, 1, 1, 0, \ldots$.
The Cauchy recursion on the other hand gives $a_0 = 1$, $a_1 - a_0 = 0$, $a_2 - a_1 + a_0 = 0$, $a_3 - a_2 + a_1 = 0$ that is, $a_{n+2} = a_{n+1} - a_n$, and we recognize the same 6-periodic sequence after computing the first eight terms as above.

Question 3

From $1 - i$ to $1 + i$, $z = 1 + ti$, $-1 \le t \le 1$ gives

$$\int_{-1}^{1} \frac{(1 - ti) i dt}{1 + t^2} = i \arctan(t)|_{-1}^{1} + \frac{1}{2} \log(1 + t^2)|_{-1}^{1} = \frac{\pi i}{2}$$

Now the other three edge integrals are obtained by multiplying z by i, i^2, i^3. Since $\frac{dz}{z}$ is unchanged under that multiplication, all contributions equal $\frac{\pi i}{2}$ and the answer is $2\pi i$.
For $\oint z^m dz$ on the unit circle, parametrize by $z = e^{i\theta}, 0 \le \theta \le 2\pi$ to get

$$\int_0^{2\pi} e^{im\theta} \cdot ie^{i\theta} d\theta = i \cdot \int_0^{2\pi} e^{im+1\theta} d\theta = i \int_0^{2\pi} (\cos(m+1)\theta + i \sin(m+1)\theta)\, d\theta$$

which vanishes if $m+1 \neq 0$. Else, we get $i\int_0^{2\pi} d\theta = 2\pi i$.

Question 4
(Without using the complex fundamental theorem of calculus) (a) Parametrize, $z = 2e^{i\theta}$ to get

$$2i \cdot \int_0^{\pi/2} (4e^{3i\theta} + 3 \cdot 2e^{2i\theta}) d\theta = 2i \left(\frac{4e^{3i\theta}}{3i} + \frac{6e^{2i\theta}}{2i} \right) \Bigg|_0^{\pi/2} = \frac{8}{3}(-i-1) + 6(-1-1) = -\frac{44}{3} - \frac{8i}{3}$$

(b) $z = 2 + t(i-1)$, $0 \leq t \leq 2$:

$$(i-1)\int_0^2 \left[(2+(i-1)t)^2 + 3(2+(i-1)t \right] dt = (i-1)\int_0^2 \left[4 + 4(i-1)t - 2it^2 + 6 + 3(i-1)t \right] dt =$$

$$(i-1)\int_0^2 \left(10 + 7(i-1)t - 2it^2 \right) dt = (i-1)\left(20 + 14(i-1) - \frac{16i}{3} \right) = -\frac{44}{3} - \frac{8i}{3}$$

(c) Have fun, but you know what you should get.

Question 5
Recalling the binomial expansion, with $\binom{m}{p} := \frac{m!}{p!(m-p)!}$, we have

$$\oint \left(z + z^{-1}\right)^{2n} \frac{dz}{z} = \oint \left(\sum_{k=0}^{2n} \binom{2n}{k} z^{2k-2n-1} \right) dz = 2\pi i \binom{2n}{n},$$

all terms other than z^{-1} contributing 1 by Question 6.
At the same time, converting to a parametrized integral by $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$ and minding that $z + z^{-1} = 2\cos\theta$ on the unit circle,

$$\oint \left(z + z^{-1}\right)^{2n} \frac{dz}{z} = 2^{2n} i \int_0^{2\pi} \cos^{2n}\theta d\theta$$

and comparing the two expressions Wallis' formula.

Question 6*
Choose an $\varepsilon > 0$. Because $\sum |a_n|$ converges, there exist an N_ε so that the sum of any finite number of terms a_k with $k > N_\varepsilon$ is $< \varepsilon$ in absolute value. Call the terms past N_ε negligible. In particular, the sum up to N_ε is no more than ε in distance away from the full $\sum a_n$.
Now let M_ε be large enough so that the b_k with $k \leq M_\varepsilon$ include all the a_n with $n \leq N_\varepsilon$ (and the correct number of times, in case there are repeated terms). Then, $\sum_{k \leq M_\varepsilon} b_n$ differs from $\sum_{n \leq N_\varepsilon} a_n$ only by negligible terms which might be included among the b_k. So

$$\left| \sum_{k \leq M_\varepsilon} b_k - \sum a_n \right| \leq \left| \sum_{k \leq M_\varepsilon} b_k - \sum_{n \leq N_\varepsilon} a_n \right| + \left| \sum_{n > N_\varepsilon} a_n \right| \leq \varepsilon + \varepsilon = 2\varepsilon$$

so that $\sum b_k$ converges to $\sum a_n$.