Solutions 4

Question 1
By our assumption g is continuously real-differentiable. The second CR equation for $g = U + iV$, $U = u_x$, $V = -u_y$ reads

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

which is the equality of the mixed partials and holds by the C^2 assumption on u. The first equation

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

is the harmonic condition.
For the last part, if $u + iv$ is holomorphic, then

$$g = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = f'$$

as desired.

Question 2
For $u = x^3 - 3xy^2$ we have $\partial u/\partial x - i\partial u \partial y = 3x^2 - 3y^2 + 6ixy = 3(x+iy)^2 = 3z^2$, which is the derivative of $f(z) = z^3$. By Problem 1, the harmonic conjugate of u is the imaginary part $3x^2y - y^3$ of z^3.
For part (b), it helps to spot $\mathrm{Re}(\bar{z}^2)$ in the numerator! So the function is

$$\mathrm{Re}\left(\frac{\bar{z}^2}{|z|^4}\right) = \mathrm{Re}\left(\frac{1}{z^2}\right)$$

and the harmonic conjugate is the imaginary part of the same.

Question 3
We can factor out $z/3$ to write the series as

$$\frac{z}{3}\sum_{n=0}^{\infty}\left(\frac{3-z}{3}\right)^n$$

and the geometric series we see is absolutely convergent for $|(3-z)/3| < 1 \Leftrightarrow |z-3| < 3$ and divergent for $|z-3| \geq 3$. It converges uniformly on any strictly smaller disk centered at $z = 3$. So our original series is also absolutely convergent in that open disk, and uniformly convergent on any smaller disk: the factor z can be bounded above by the constant 6 so does not spoil the absolute or uniform convergence.
However, the original series also converges at $z = 0$, because all the terms vanish.

Question 4

(a) $|x| \le 1$ because $\sum \frac{1}{n^2}$ converges; (b) $|x| < 2$ pretty clear; (c) any x, ratio test or comparison test with $\sum \frac{y^n}{n!}$ by setting $y = \sqrt{|x|}$; (d) $x = 0$ only, ratio test.

Question 5

We have $\lim_{n\to\infty} (n^n)^{1/n} = \lim_{n\to\infty} n = \infty$, so $R = 0$ by Hadamard.
Now $\lim_{n\to\infty} \left(n^2\right)^{1/n} = \left(\lim_{n\to\infty} n^{1/n}\right)^2 = 1$ because $\frac{1}{n}\log n \to 0$, by l'Hôpital say, so $R = 1$. We write

$$\sum_{n=0}^{\infty} n^2 z^n = \sum_{n=0}^{\infty} n(n-1)z^n + \sum_{n=0}^{\infty} nz^n = z^2 \sum_{n=0}^{\infty} n(n-1)z^{n-2} + z\sum_{n=0}^{\infty} nz^{n-1} =$$

$$= z^2 \frac{d^2}{dz^2} \sum_{n=0}^{\infty} z^n + z\frac{d}{dz}\sum_{n=0}^{\infty} z^n = z^2 \frac{d^2}{dz^2}\frac{1}{1-z} + z\frac{d}{dz}\frac{1}{1-z} = \frac{z+z^2}{(1-z)^3}$$

Remark: You can also get the radii of convergence by using the ratio test.

Question 6

Using the triangle inequality, it suffices to show that we can keep $|z^k \sum_{n\ge k} a_n z^{n-k}| < |a_0|$, for small $|z|$. If r is the radius of convergence of the series, then $|a_n|\rho^n \to 0$ for any $\rho < r$. Keeping $|z| < \rho/2$ say, we get $|a_n z^{n-k}| < C/2^n$ for all $n \ge k$ and some constant C. Then, $\sum_{n\ge k} |a_n z^{n-k}| < 2C$. Confine now z so that, in addition to the above, $2C \cdot |z|^k < |a_0|$.

Question 7

$$\frac{1-z}{1+z} \cdot \frac{1+z}{1-z} = 1,$$

and that product has infinite radius of convergence. But the series expansions for each of the two factors near $z = 0$ (which you can derive easily from the geometric series) has radius of convergence 1.