Solutions 3

Question 1
The Jacobian matrix for the change from Cartesian to polar coordinates is

$$J = \begin{bmatrix} \partial r/\partial x & \partial\theta/\partial x \\ \partial r/\partial y & \partial\theta/\partial y \end{bmatrix} = \begin{bmatrix} x/r & -y/r^2 \\ y/r & x/r^2 \end{bmatrix}$$

and so we get from the 2-dimensional real chain rule

$$\begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix} = J \cdot \begin{bmatrix} \partial u/\partial r \\ \partial u/\partial\theta \end{bmatrix}, \qquad \begin{bmatrix} \partial v/\partial x \\ \partial v/\partial y \end{bmatrix} = J \cdot \begin{bmatrix} \partial v/\partial r \\ \partial v/\partial\theta \end{bmatrix}$$

The Cartesian CR equations

$$\begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix} = \begin{bmatrix} \partial v/\partial y \\ -\partial v/\partial x \end{bmatrix}$$

become in polar coordinates (we switch the rows of J and change the sign of the bottom row)

$$\begin{bmatrix} x/r & -y/r^2 \\ y/r & x/r^2 \end{bmatrix} \cdot \begin{bmatrix} \partial u/\partial r \\ \partial u/\partial\theta \end{bmatrix} = \begin{bmatrix} y/r & x/r^2 \\ -x/r & y/r^2 \end{bmatrix} \cdot \begin{bmatrix} \partial v/\partial r \\ \partial v/\partial\theta \end{bmatrix}$$

which we can solve to

$$\begin{bmatrix} \partial u/\partial r \\ \partial u/\partial\theta \end{bmatrix} = \begin{bmatrix} 0 & 1/r \\ -r & 0 \end{bmatrix} \cdot \begin{bmatrix} \partial v/\partial r \\ \partial v/\partial\theta \end{bmatrix} = \begin{bmatrix} 1/r \cdot \partial v/\partial\theta \\ -r \cdot \partial v/\partial r \end{bmatrix},$$

so finally

$$r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial\theta}, \qquad \frac{\partial u}{\partial\theta} = -r\frac{\partial v}{\partial r}.$$

Question 2
Let as usual $f = u + iv$. We know that $f'(z) = p + iq$, where the Jacobian matrix of f is

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$$

so $f' = u_x + iv_x$. But

$$\partial f/\partial z = \frac{1}{2}(\partial_x - i\partial_y)(u + iv) = \frac{u_x + v_y}{2} + i\frac{v_x - u_y}{2} = u_x + iv_x$$

by CR, as needed. To check Cauchy-Riemann for $f' = \partial u/\partial x + i\partial v/\partial x$, use the CR equations for u, v and the equality of the mixed partials to get

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}, \\ \frac{\partial^2 u}{\partial y \partial x} &= \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}. \end{aligned}$$

Question 3
Horizontal lines require the imaginary part to be constant, and the curves where y/x is constant are straight lines through the origin. Vertical lines have the real part $x^2 + y^2$ constant, which describes circles centered at 0. So the polar coordinate grid is mapped to the Cartesian coordinate grid.
Since $\partial u/\partial x = 2x$ and $\partial v/\partial y = 1/x$, CR fails for most x. The curve $t \mapsto (x, y) = (t, t)$ gets transformed into the curve $t \mapsto (2t^2, 1)$ and $t \mapsto (1, t)$ gets mapped to the horizontal parabola $t \mapsto (t^2 + 1, t)$. The original curves meet at $(x, y) = (1, 1)$ with angle $\pi/4$ but the angle of the second set of curves at their meeting point $(2, 1)$ is not $\pi/4$, as the first curve is horizontal and the slope of the second velocity vector is $1/2$.
For the last part, note that $x^2 + y^2$ is not harmonic so it cannot be the real part of a holomorphic function.

Question 4
The function is clearly real-differentiable. With $u = e^x \cos y$ and $v = e^x \sin y$, we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos y, & \frac{\partial u}{\partial y} &= -e^x \sin y, \\ \frac{\partial v}{\partial x} &= e^x \sin y, & \frac{\partial v}{\partial y} &= e^x \cos y, \end{aligned}$$

and Cauchy-Riemann implies complex-differentiability. The derivative is then $\partial u/\partial x + i\partial v/\partial x = e^x(\cos y + i \sin y)$ and agrees with the function itself. The last check follows from the addition formulae for sin and cos.

Question 5
Note that $\pi/4 = 2\pi/8$ and $\pi/2 = 4\pi/8$, so by symmetry, $\arg(\exp(\pi i/4) + \exp(\pi i/2)) = 3\pi i/8$. So $\tan(3\pi/8) = \mathrm{Im}/\mathrm{Re}$ of that sum which is $\left(1 + \sqrt{2}/2\right)/(\sqrt{2}/2) = 1 + \sqrt{2}$.
For the second part, note that $(2+i)(3+i) = 5+5i$ and so $\arg(2+i)+\arg(3+i) = \arg(5+5i) = \pi/4$; but these arguments are $\tan^{-1}(1/2)$ and $\tan^{-1}(1/3)$, respectively.

Question 6
Yes (it's $\exp(iz)$); No; No; Yes (it's $z \cdot \exp(z)$). A Cauchy-Riemann computation settles them all, good luck.

Question 7

$$\begin{aligned} \cos z = \cos(x + iy) = \cos x \cos(iy) - \sin x \sin(iy) = \cos x \cosh(-y) \\ - \sin x \sinh(-y)/i = \cos x \cosh(-y) - i \sin x \sinh y, \end{aligned}$$

similarly for sin. The absolute values now come out by summing the squares of real and imaginary parts:

$$\begin{aligned} |\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = \cos^2 x + \cos^2 x(\cosh^2 y) - 1 + \sin^2 x \sinh^2 y = \\ \cos^2 x + \cos^2 x \sinh^2 y + \sin^2 x \sinh^2 y = \cos^2 x + \sinh^2 y, \end{aligned}$$

and similarly for $|\sin z|^2$.

Question 8
Parametrize the lines as $u + it$, fixed u, or $t + iv$, fixed v, and use the formulas from the previous exercise; for instance, the image of a vertical line $\cos(u + it) = \cos u \cosh t - i \sin u \sinh t$ is a

centered, parametized half-hyperbola, satisfying the equation

$$\left(\frac{x}{\cos u}\right)^2 - \left(\frac{y}{\sin u}\right)^2 = 1.$$

Something special happens when $\sin u = 0$, when you just sweep out the rays $(-\infty, -1]$ or $[1, \infty)$ twice, from ∞ and back: the hyperbola gets flattened out to the two rays, and when $\cos u = 0$, when you sweep out the imaginary axis.