Solutions 2

Question 1

The map $z \mapsto a \cdot z$, for fixed $a = \cos\theta + \mathrm{i}\sin\theta$ is a rotation by angle θ. The map $z \mapsto z + b$ is a translation by the vector b. For the map $z \mapsto az + b$, which is a composition of a rotation and a tranlation, we have two possibilities:

1. The angle $\theta \neq 0 \pmod{2\pi\mathbf{Z}}$. The composition is then a rotation about some center in the plane. To find the center c, we are looking for the complex number c fixed by the transformation, so that $c = ac + b$, which forces $c = b/(1-a)$. To verify our conclusion, note that the rotation about c y angle θ must be the map

$$z \mapsto c + (z - c) \cdot (\cos\theta + \mathrm{i}\sin\theta) = \frac{b}{1-a} + za - ca = \frac{b}{1-a} - \frac{ba}{1-a} + za = b + za.$$

2. The angle $\theta = 0 \pmod{2\pi\mathbf{Z}}$. Then the map is a pure translation by b.

Question 2

We need de Moivre's formulae: $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$, so the cube roots are (keeping in mind to add multiples of $2\pi/3$)

$$\begin{aligned}
\cos\frac{\pi}{6} + i\sin\frac{\pi}{6} &= \frac{\sqrt{3}}{2} + \frac{i}{2}\\
\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} &= -\frac{\sqrt{3}}{2} + \frac{i}{2}\\
\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6} &= -i.
\end{aligned}$$

For the second part, $1 + i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ so $(1+i)^{1000} = 2^{500}$, because $1000 \cdot \frac{\pi}{4}$ is a multiple of 2π.

Question 3

The four complex roots of -1 are $\frac{1}{\sqrt{2}}(\pm 1 \pm i)$, with the four independent choices of signs. So

$$x^4 + 1 = (x + \frac{1+i}{\sqrt{2}})(x + \frac{1-i}{\sqrt{2}})(x - \frac{1+i}{\sqrt{2}})(x - \frac{1-i}{\sqrt{2}})$$

which gives $(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$ after combining the first two and the last two factors.

Question 4

Check that $\bar{a}b + \bar{b}c + \bar{c}a$ changes by a real number a, b and c are simultaneously replaced by $a + z, b + z$ and $c + z$ (translation by the complex number z). Since the area is also invariant under translation, it suffices to verify the formula when $a = 0$. (Translate all by $-a$). Note further that $\mathrm{Im}(\bar{b}c)$ is unchanged when b and c are both multiplied by the same complex number

α of unit modulus. This represents a rigid rotation, and it also leaves the area invariant. We choose α so that its argument is $-\arg b$, and then b becomes positive real.

The triangle now has the first vertex at 0, the second at the real number b, the third at a complex number c. Its area is $\frac{1}{2}$ base$\times$height, which is half $b \times \mathrm{Im}(c)$. Hence the formula given is always valid.

Question 5

In coordinates x, y, we have

$$f(z) = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}$$

and as x, y and $x^2 + y^2 = r^2$ are continuously differentiable and the last one is non-zero in its domain it follows that f is continuously real-differentiable. We have

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{-x^2 + y^2}{r^4}, & \frac{\partial u}{\partial y} &= \frac{-2xy}{r^4}, \\ \frac{\partial v}{\partial x} &= \frac{2xy}{r^4}, & \frac{\partial v}{\partial y} &= \frac{-x^2 + y^2}{r^4}, \end{aligned}$$

confirming the Cauchy-Riemann equations.

To check differentiability from the complex definition, note that

$$f(z) - f(z_0) = \frac{1}{z} - \frac{1}{z_0} = \frac{z_0 - z}{z z_0},$$

and dividing by $z - z_0$ gives

$$\lim \frac{f(z) - f(z_0)}{z - z_0} = -\lim \frac{1}{z z_0} = -\frac{1}{z_0^2}$$

since $\frac{1}{z}$ has already been shown to be continuous.

Question 6

(a,b) Nowhere, Cauchy-Riemann fails everywhere (c) Check CR via $\partial/\partial\bar{z}$: $\partial f/\partial\bar{z} = z^2 - 1$ vanishes only at $z = \pm 1$ so f is complex-differentiable only at those points; (d) this is $1/z$ so it's holomorphic away form 0 but not even continuous there.

Question 7

(a) From Cauchy-Riemann, all partials of the real and imaginary parts of f vanish, so both $\mathrm{Re} f$ and $\mathrm{Im} f$ are constant. (b) $v = 0$ so the partials of u vanish everywhere so f is constant. (c) Here is a general argument: if f is holomorphic and f' fails to vanish at some z_0, then the Jacobian matrix is invertible there and it follows that f maps a small neighbourhood of z_0 onto a small neighbourhood of $f(z_0)$. In particular, if f' is not identically zero, the image of f cannot be contained in any regular curve, such as $|w| = \mathit{const}$ (d) It follows that $f + \bar{f}$ and $i(f - \bar{f})$ are both holomorphic; but they are real-valued so they are both constant by a and it follows that f is constant.

Question 8

Good luck. (Ask me if you wish).