Solutions 11

Question 1 See attached pdf

Question 2
I'm afraid we were missing the assumption $k > 1$.
It is easy to show that the map takes the circle to the ellipse: if $z = e^{i\theta}$ and $k = e^{\alpha}$, then

$$w = k^{-1}e^{i\theta} + ke^{-i\theta} = (k^{-1} + k)\cos\theta + i(k^{-1} - k)\sin\theta,$$

the standard parametrization of an ellipse if $k < 1$, and the clockwise parametrization if $k > 1$. Let now $k > 1$ and compose with the map $w \mapsto 1/w$ to get a *holomorphic* map inside the unit disk:

$$(k^{-1}z + kz^{-1})^{-1} \text{ is singular } \Leftrightarrow z = \pm ik \Leftrightarrow |z| > 1$$

This takes the unit circle bijectively to the clockwise parametrized ellipse, and the argument principle now ensures that the interior of the disk maps bijectively to the interior of the ellipse. Try to show similarly that, if $k < 1$, then the exterior of the disk plus $\{\infty\}$ maps to the exterior of the ellipse.

Question 3
The easy way: $\cos 2\theta = \cos^2\theta - \sin^2\theta = x^2 - y^2 = \mathrm{Re}(z^2)$, which is harmonic on the entire plane. By the Poisson method (mind, $z = e^{i\varphi}$):

$$\frac{1-r^2}{2\pi}\int_0^{2\pi}\frac{\cos(2\varphi)d\varphi}{1-2r\cos(\theta-\varphi)+r^2} = \frac{1-r^2}{4\pi}\oint_C \frac{(z^4+1)dz}{iz^3(1-r(e^{i\theta}z^{-1}+e^{-i\theta}z)+r^2)}$$

The denominator factors as $iz^2(z - re^{i\theta})(1 - re^{-i\theta}z)$, with zeroes inside the unit circle at $z = 0$ and $z = re^{i\theta}$. The residues at 0 and $re^{i\theta}$ are

$$ir^{-2}e^{-2i\theta}(1+r^2) \text{ and } (-i)r^{-2}e^{-2i\theta}\frac{r^4e^{4i\theta}+1}{1-r^2}$$

so we get from the residue formula

$$\frac{1-r^2}{4\pi}\cdot 2\pi i\cdot\left(ir^{-2}e^{-2i\theta}(1+r^2) + (-i)r^{-2}e^{-2i\theta}\frac{r^4e^{4i\theta}+1}{1-r^2}\right) = r^2\frac{e^{2i\theta}+e^{-2i\theta}}{2} = r^2\cos 2\theta$$

Question 4
Translating everything down by x, we reduce to the case when $x = 0$. Because of claim (2) below, we may also shift f by the constant function with value $f(0)$, and reduce our statement to the case when $f(0) = 0$. I claim that:

1. The function $y/\pi(t^2 + y^2)$ is positive for all $y > 0$;
2. Its integral over t is identically 1, for all $y > 0$;

3. For any fixed $\varepsilon > 0$, there exists $\delta > 0$ so that for $0 < y < \delta$,

$$\int_{-\varepsilon}^{+\varepsilon} \frac{ydt}{\pi(t^2+y^2)} > 1 - \varepsilon$$

(and hence by (2) the integral outside is less than ε.)

Claim 1 is obvious, 2 and 3 are seen from the explicit antiderivative $\pi^{-1}\arctan t/y$: for instance, in 3, after changing variables to t/y, the integral is $\frac{2}{\pi}\arctan(\varepsilon/y)$ which for fixed $\varepsilon > 0$ approaches 1 as $y \to 0$.
Now

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yf(t)dt}{t^2+y^2} = \frac{1}{\pi}\int_{-\infty}^{-\varepsilon} \frac{yf(t)dt}{t^2+y^2} + \frac{1}{\pi}\int_{\varepsilon}^{\infty} \frac{yf(t)dt}{t^2+y^2} + \frac{1}{\pi}\int_{-\varepsilon}^{\varepsilon} \frac{yf(t)dt}{t^2+y^2}$$

and if $0 < y < \delta$,

$$\left|\frac{1}{\pi}\int_{-\infty}^{-\varepsilon} \frac{yf(t)dt}{t^2+y^2} + \frac{1}{\pi}\int_{\varepsilon}^{\infty} \frac{yf(t)dt}{t^2+y^2}\right| < \frac{1}{\pi}\int_{-\infty}^{-\varepsilon} \frac{Mydt}{t^2+y^2} + \frac{1}{\pi}\int_{\varepsilon}^{\infty} \frac{Mydt}{t^2+y^2} < M\varepsilon \quad (M = \sup f)$$

Choose now $\eta > 0$ and choose $0 < \varepsilon < \eta$ so that $|f(t)| < \eta$ for $|t| < \varepsilon$, possible because $f(0) = 0$ and by the continuity of f. Since

$$\left|\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yf(t)dt}{t^2+y^2}\right| < M\varepsilon + \left|\frac{1}{\pi}\int_{-\varepsilon}^{\varepsilon} \frac{yf(t)dt}{t^2+y^2}\right| < M\varepsilon + \left|\frac{1}{\pi}\int_{-\varepsilon}^{\varepsilon} \frac{y\eta dt}{t^2+y^2}\right| < M\eta + \eta = (M+1)\eta$$

provided $0 < y < \delta$ as above. So we have shown that $\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yf(t)dt}{t^2+y^2} \to 0 = f(0)$ as $y \to 0$.
Remark: This shows the pointwise convergence $\Phi(x,y) \to f(x)$ in the question. It is not difficult to improve the argument to get *local uniform convergence*: the only dependence on f was the choice of ε given η, which is controlled by the continuity properties of f. Every continuous function on a closed, bounded interval is uniformly continuous, so a ε can be chosen to work on every bounded interval.

Question 5
Using the kernel above or by eyeballing in terms of the argument function,

$$1 - \frac{1}{\pi}\left(\mathrm{Arg}(z-1) + \mathrm{Arg}(z+1)\right)$$

Question 6
Use the Laplacian in polar coordinates centered at the point in question; the harmonic condition on u is

$$u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta} = 0$$

Since u depends only on r, the last term vanishes and we have $u_{rr} + r^{-1}u_r = 0$. Let $f = ru_r$, then $f_r = u_r + ru_{rr} = 0$ so f is a constant A and then $u_r = A/r$ and u is as advertised.

Question 7
We choose the root which is positive on the real axis above 1. By following z on a large semi-circle in the upper half-plane, we find that the argument of $z^2 - 1$ reaches 2π by the time we reach the (large) negative axis, so the square root takes negative values there.
Writing $z = x + iy$, $w = u + iv$ and solving for $w^2 = z^2 - 1$, we get

$$u^2 = \frac{x^2 - y^2 - 1}{2} \pm \frac{1}{2}\sqrt{(x^2+y^2-1)^2 + 4y^2}, \quad v = \frac{xy}{u}$$

This can only vanish when $xy = 0$, when we'll need to work out the value of v differently.
Meanwhile, positivity of u^2 requires the positive square root. (Note that the term inside the root vanishes only when $x = \pm 1$, $y = 0$.) That means that on the interval $-1 < x < 1$ we want to get $(1 - x^2)$, and so $u = 0$ there whereas $u = \pm\sqrt{x^2 - 1}$, with sign as discussed, outside that interval, and in that case $v = 0$.
The sign of u cannot change without u vanishing, which only happens for imaginary z (and the real interval $[-1, 1]$): this pins the sign of u uniquely, as the same as that of x, away from the imaginary axis. On that axis, w is purely imaginary so $u = 0$ and we have pinned u uniquely as a continuous function of x, y.
v is determined by the second equation away from the imaginary axis; near there, we can find it from $v^2 = u^2 + 1 + y^2 - x^2$ as a continuous function of x, y; we choose the positive square root to match $v \geq 0$ in both quadrants.
To show that the map is injective and surjective, we need to solve uniquely for x, y in terms of u, v. This is done by reviewing the discussion with the equations

$$x^2 = \frac{u^2 - v^2 + 1}{2} + \frac{1}{2}\sqrt{(u^2 + v^2 + 1)^2 - 4v^2}, \quad y = \frac{uv}{x}$$

and choosing the appropriate signs for x. The ambiguity in the equation for y at $x = 0$ is addressed again by $v^2 - u^2 + x^2 - 1 = y^2$, uniquely solvable with positive y when $v \geq 1$. (Recall that the imaginary points with $v < 1$ are in the image of the real interval $[-1, 1]$ and not the imaginary axis.)